

EVOLUTION OF APPROXIMATION SPACES IN ROUGH SETS: FROM RELATIONAL SYSTEMS TO COMPLEX ADAPTIVE SYSTEMS

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AGENDA

- Rough Sets based on relational systems
- Motivations for developing a computing model that creates the basis for designing and analyzing complex intelligent systems (ISs), i.e., systems with behavior that can be difficult to separate from interactions with complex real-world phenomena and/or complex adaptive systems.
- From Granular Computing (GrC) to Interactive Granular Computing (IGrC)
- Rough Sets (RS) in IGrC
 - Dynamic approximation spaces
 - Approximate problem solving grounded in IGrC
- Summary

ROUGH SETS BASED ON RELATIONAL SYSTEMS

THE PAWLAK MODEL OF ROUGH SETS

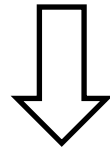
THE PAWLAK MODEL OF ROUGH SETS

$$AS=(U, r)$$

U - finite universe of objects

r - equivalence (indiscernibility) relation in U

$X \subseteq U$ – *concept*



AS -approximations of X

$$LOW(X, AS) = \{x \in U: [x]_r \subseteq X\}$$

$$UPP(X, AS) = \{x \in U: [x]_r \cap X \neq \emptyset\}$$

$$Bd(X, AS) = UPP(X, AS) \setminus LOW(X, AS)$$

THE PAWLAK MODEL OF ROUGH SETS OVER INFORMATION SYSTEMS

Information system (data table)

$$IS = (U, A)$$

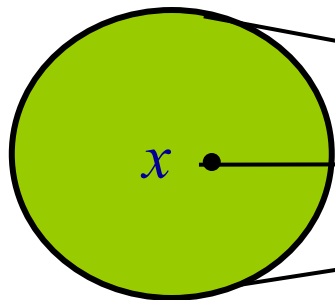
$$U = \{x_1, \dots, x_n\}, A = \{a_1, \dots, a_m\}, a_i: U \rightarrow V_{a_i}$$

$$B \subseteq A$$

B – indiscernibility relation of IS

$$xIND(B)y \text{ iff } Inf_B(x) = Inf_B(y)$$

$$N_B(x) = [x]_{IND(B)} = [x]_B = \{y \in U: xIND(B)y\}$$



neighborhood of x
elementary granule

$$U/B = \{[x]_B: x \in U\}$$

STEMS

B

	a_1	a_2	$\dots$	a_m
x_1	v_1	v_2	$\dots$	v_m
	$\dots$	$\dots$	$\dots$	$\dots$

$u = Inf_B(x)$ signature of x

$$Inf_B(x) = Inf_B(y)$$

$\uparrow$
 τ

tolerance or similarity ⁶

THE PAWLAK MODEL OF ROUGH SETS OVER INFORMATION SYSTEMS

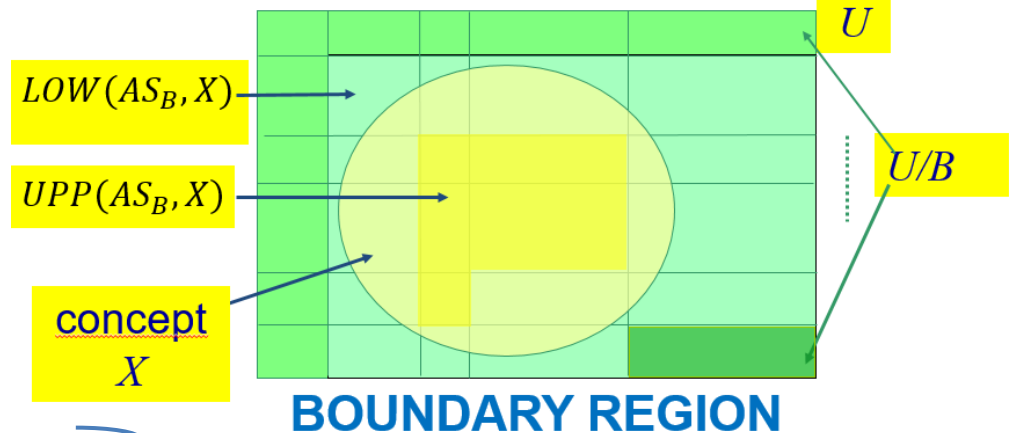
LOWER AND UPPER APPROXIMATION

	a_1	a_2	...	a_m
x_1	v_1	v_2	...	v_m
	...	...	...	...

$$X \subseteq U, B \subseteq A, AS_B = (U, IND(B))$$

$$LOW(AS_B, X) = \cup \{Y \in U/B : Y \subseteq X\}$$

$$UPP(AS_B, X) = \cup \{Y \in U/B : Y \cap X \neq \emptyset\}$$



$$Bd(AS_B, X) = UPP(AS_B, X) \setminus LOW(AS_B, X)$$

↑
computational building blocks
for concept approximation in
the rough set approach

- indiscernibility classes of (subsets of) attributes
- partitions defined by attributes
- partitions defined by subsets of attributes
- granules defined by calculi over the above granules
- ...

THE PAWLAK MODEL OF ROUGH SETS OVER INFORMATION SYSTEMS

$IS=(U, A)$ – information system; $B \subseteq A$

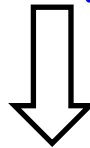
$X \subseteq U$ – concept



$AS_B = (U, \{r_a\}_{a \in B})$ – multi-relational approximation space

$x r_a y$ iff $a(x) = a(y)$

$$r_B = \bigcap_{a \in B} r_a$$



AS_B -approximations of X

$$LOW(X, AS_B) = \{x \in U: [x]_B \subseteq X\}$$

$$UPP(X, AS_B) = \{x \in U: [x]_B \cap X \neq \emptyset\}$$

$$Bd(X, AS_B) = UPP(X, AS_B) \setminus LOW(X, AS_B)$$

NETWORK OF APPROXIMATION SPACES LINKED BY INTERFACES: THE PAWLAK ROUGH SET MODEL OVER DECISION SYSTEMS

$$G_R = \{g_x : x \in U\}$$

$$G_{R_d} = \{g'_y : y \in U\}$$

$R \subseteq U \times U$ – equivalence relation

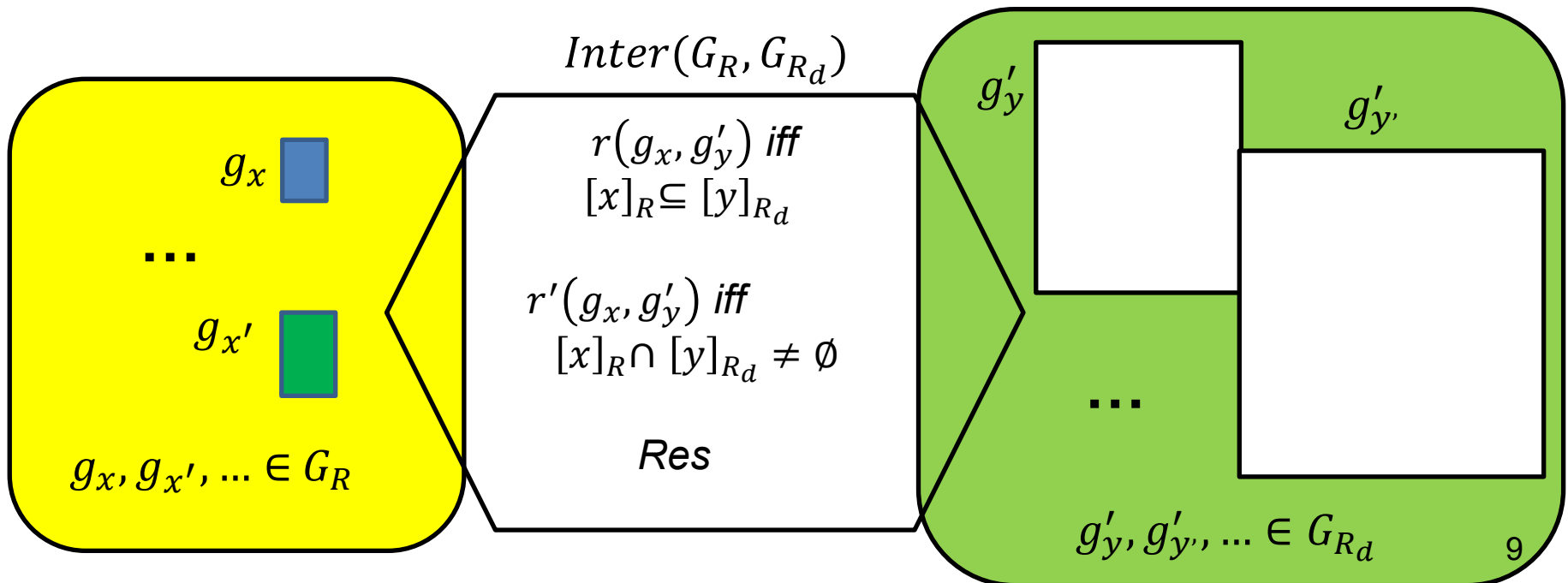
$R_d \subseteq U \times U$ – equivalence relation

$$g_x = (f([x]_R), [x]_R), x \in U$$

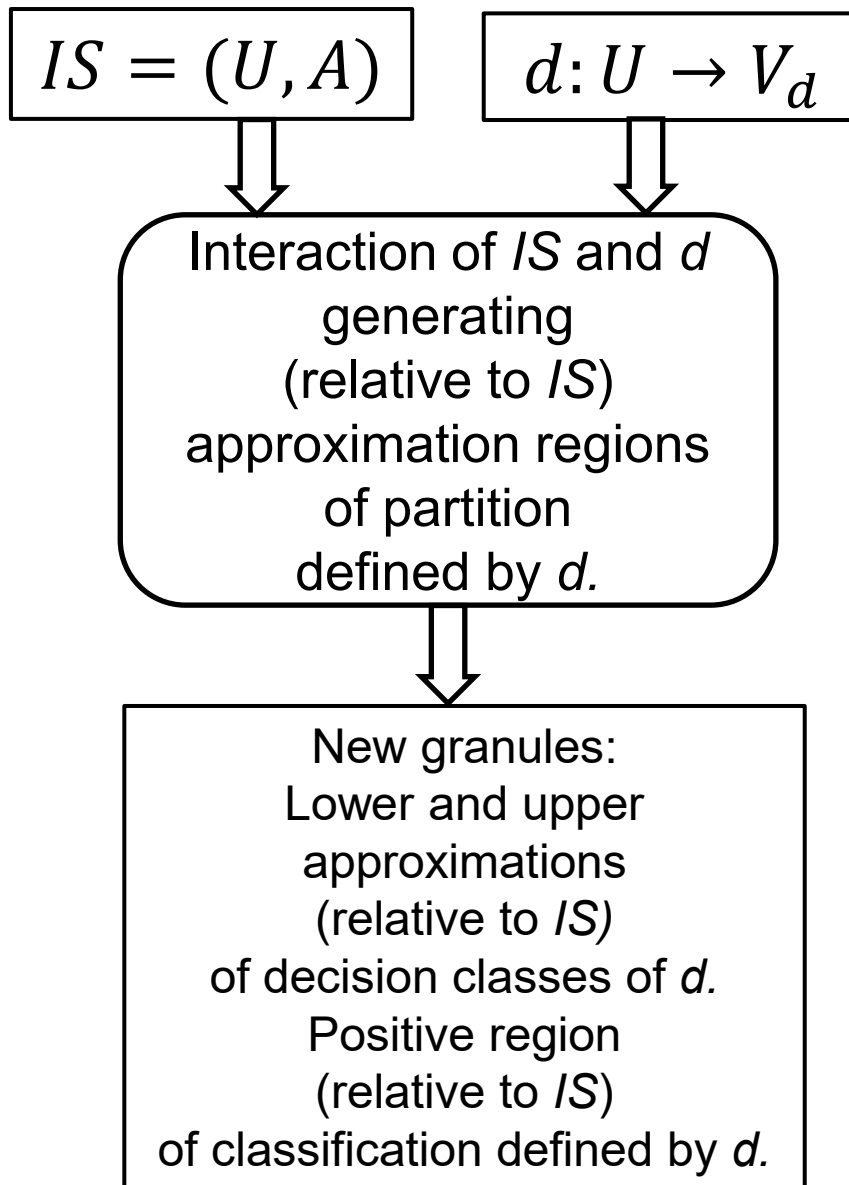
$$g'_y = (h([y]_{R_d}), [y]_{R_d}), y \in U$$

$f: U/R \rightarrow \{1, \dots, \text{card}(U/R)\}$ – bijection

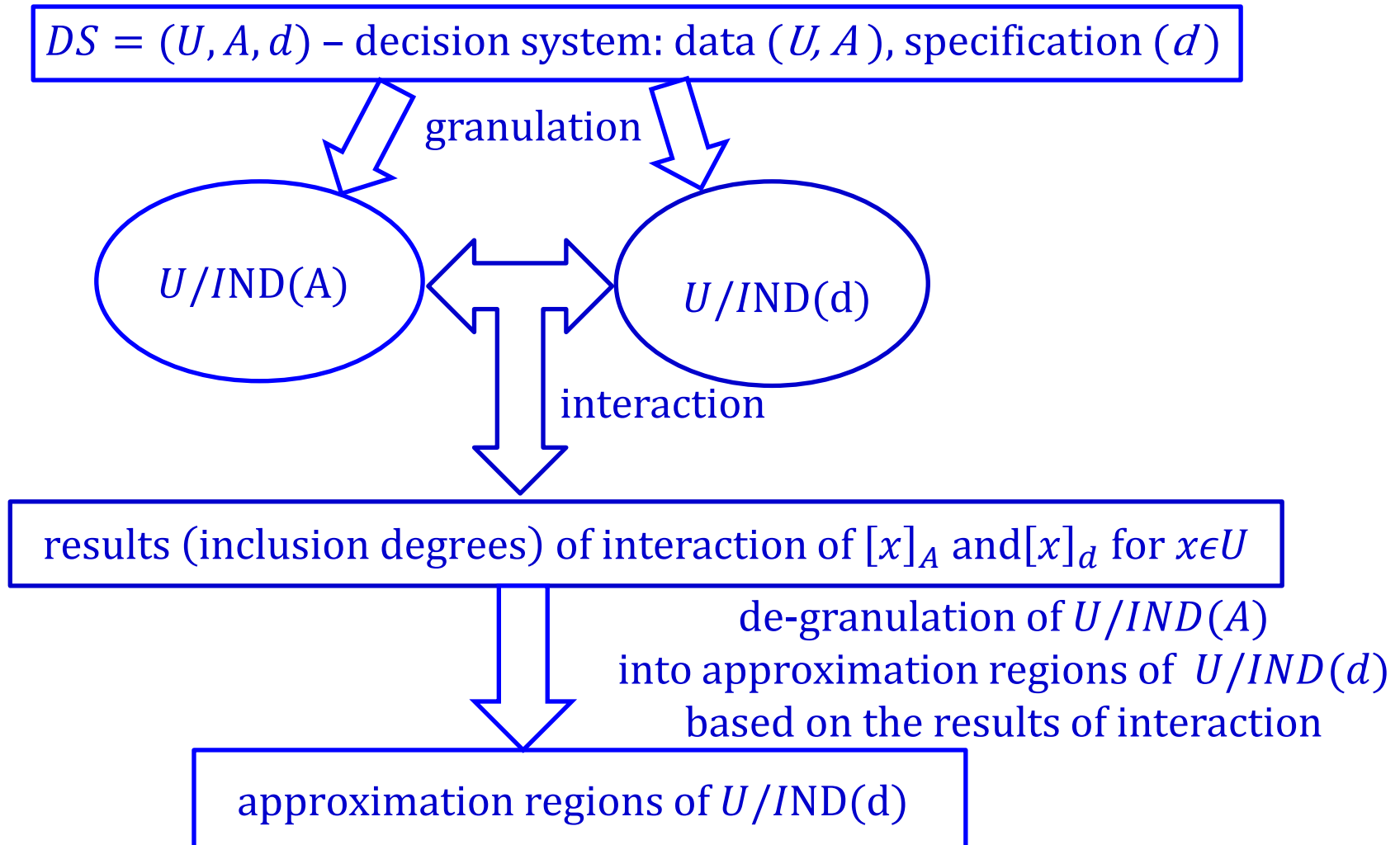
$h: U/R_d \rightarrow \{1, \dots, \text{card}(U/R_d)\}$ – bijection



INFORMATION GRANULES GENERATED BY INTERACTION IN THE PAWLAK ROUGH SET MODEL



GENERAL SCHEME OF APPROXIMATION OF GRANULES STEERED BY INTERACTIONS OF GRANULES GENERATED THROUGH MULTILEVEL GRANULATION AND DE- GRANULATION PROCESSES: ILLUSTRATIVE EXAMPLE



OPTIMIZATION PROBLEMS

OPTIMIZATION PROBLEMS: EXAMPLE

FOR GIVEN

$GRel = (U, \{r_i\}_{i \in I})$, where $r_i \subseteq U \times U$ - indiscernibility relation

I - can be finite or infinite

$P_\omega(I)$ - the family of finite non-empty subsets of I

$Al(\{r_i\}_{i \in J})$ - learning algorithm parameterized by $\{r_i\}_{i \in J}$ (where $J \in P_\omega(I)$)
returning a classifier $Cl_{Al}(\{r_i\}_{i \in J})$

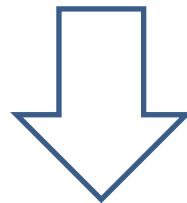
from a decision system $(U, \bigcap_{i \in J} r_i, d)$,

where $d: U \rightarrow V_d$ and V_d is a set of decision values

Q - quality measure defined over $Cl_{Al}(\{r_i\}_{i \in J})$ for $J \in P_\omega(I)$
with values in $[0, 1]$

$tr \in [0, 1]$ - quality threshold

k - bound on cardinality of subsets of I



FIND

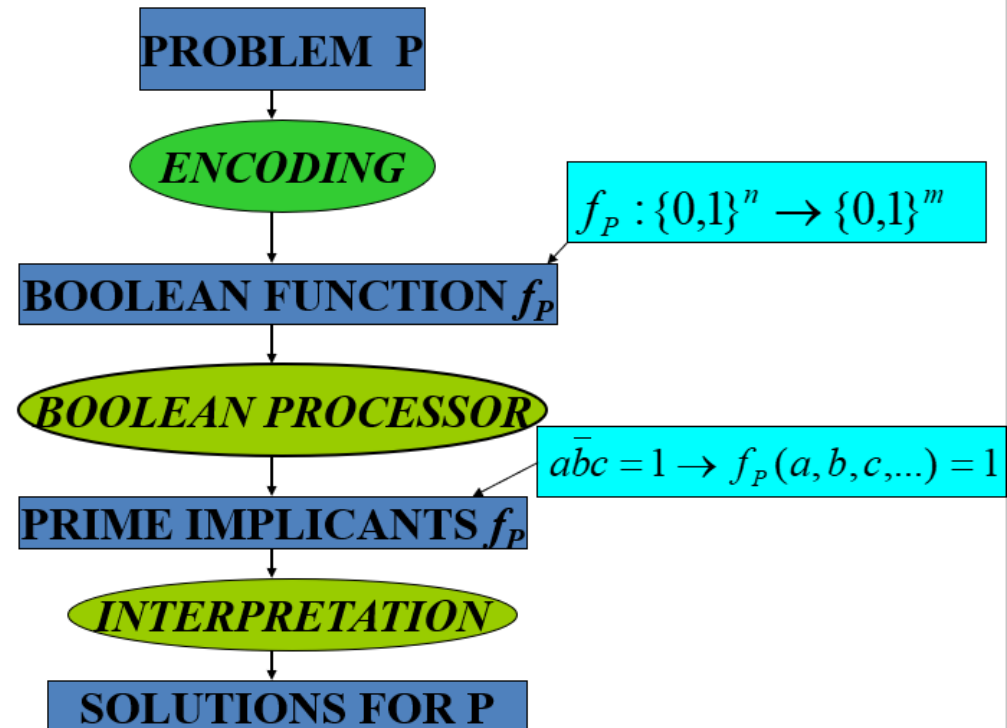
$J \in P_\omega(I)$ with quality $Q(Cl_{Al}(\{r_i\}_{i \in J})) \geq tr$ and minimal
cardinality $|J| \leq k$, if such J exists.

OPTIMIZATION PROBLEMS

BOOLEAN REASONING

George Boole (1815-1864)

- feature selection
- data reduction
- discretization
- symbolic value grouping
- decision rules
- ...



Z. Pawlak, A. Skowron: *Rough Sets and Boolean Reasoning*. Inf. Sci. 177(1) 41-73 (2007)

GRANULAR COMPUTING (GrC)

Information granules: foundational building blocks of data, representing specific, detailed entities that group primary information (like numbers) based on their similarities, proximity, or functional equivalence. Embedded in the abstract space.

- Operations on information granules:
granulation (*decomposition*, top-down):
splitting into more specific granules
- *de-granulation* (*aggregation*, bottom up):
combining into more general granules

GRANULAR CALCULI

GRANULAR CALCULUS

$GA = (At, O)$ - granular algebra

$Gr(GA)$ - set of granules generated by GA

$GAC = (At, O, C); C \subseteq Gr(AC)$ - context

$GRel$ – granular relational system

$GRel = (C, R), R = \{r_i\}_i$

L – language

$\alpha(x_1, \dots, x_k)$ - formula

Int – interpretation of L in $GRel$

$Int(\alpha(x_1, \dots, x_k)) \subseteq C^k$

Sel - selection of the set of relevant granules (formulas of L) as computational building blocks for cognition evaluated on the basis of *Quality* measure

should be discovered and adapted from data and knowledge bases in cooperation with users, experts, chatbots; relations discovered using both the abstract and physical semantics

APPROXIMATION SPACES GROUNDED IN GRANULAR RELATIONAL SYSTEMS (GRel) AND LANGUAGES WITH INTERPRETATION IN GRel

$$GRel = (U, \{r_i\}_{i \in I})$$

Contains c-granules provided by the lower hierarchical level or from interactions with the physical space (e.g., sensory measurements).

Contains relations over U (including also constants and functions) discovered by the control of c-granule using abstract and physical semantics.

L with interpretation in $GRel$

L -language, discovered through the control of c-granule and interpreted in GRel, is used to define the syntax and semantics of newly discovered c-granules.

These c-granules serve on the next hierarchical level as computational building blocks for cognition, supporting the approximation of concepts, particularly target concepts at the top hierarchical level.

The quality of these new granules is evaluated using specific measures.

These new c-granules also enable the induction of rules stored in interfaces linking hierarchical levels, making it possible to infer the properties of these c-granules from the granules from which they were constructed.

LINKING GRANULAR NETWORKS BY INTERFACES

PRODUCTS WITH CONSTRAINTS OF ALREADY CONSTRUCTED GRANULAR NETWORKS

GN_1 granular network

- elementary (atomic) granules
- transformations for constructing granules
- methods (algorithms) for generation new granules, modifying existing ones and reasoning about computations over them
- can have nested structure
- ...

Interface

$Inter(GN_1, GN_2)$

with knowledge for reasoning about properties of granules from higher hierarchical levels on the basis of granules from the lower layers:

- relations between granules from GN_1 and GN_2
- transformations of granules from GN_1 to GN_2
- rules for reasoning about properties of granules from GN_2 on the basis of properties of granules from GN_1 transformed to granules from GN_2
- methods for generation of new samples of granules in GN_1 and GN_2
- ...

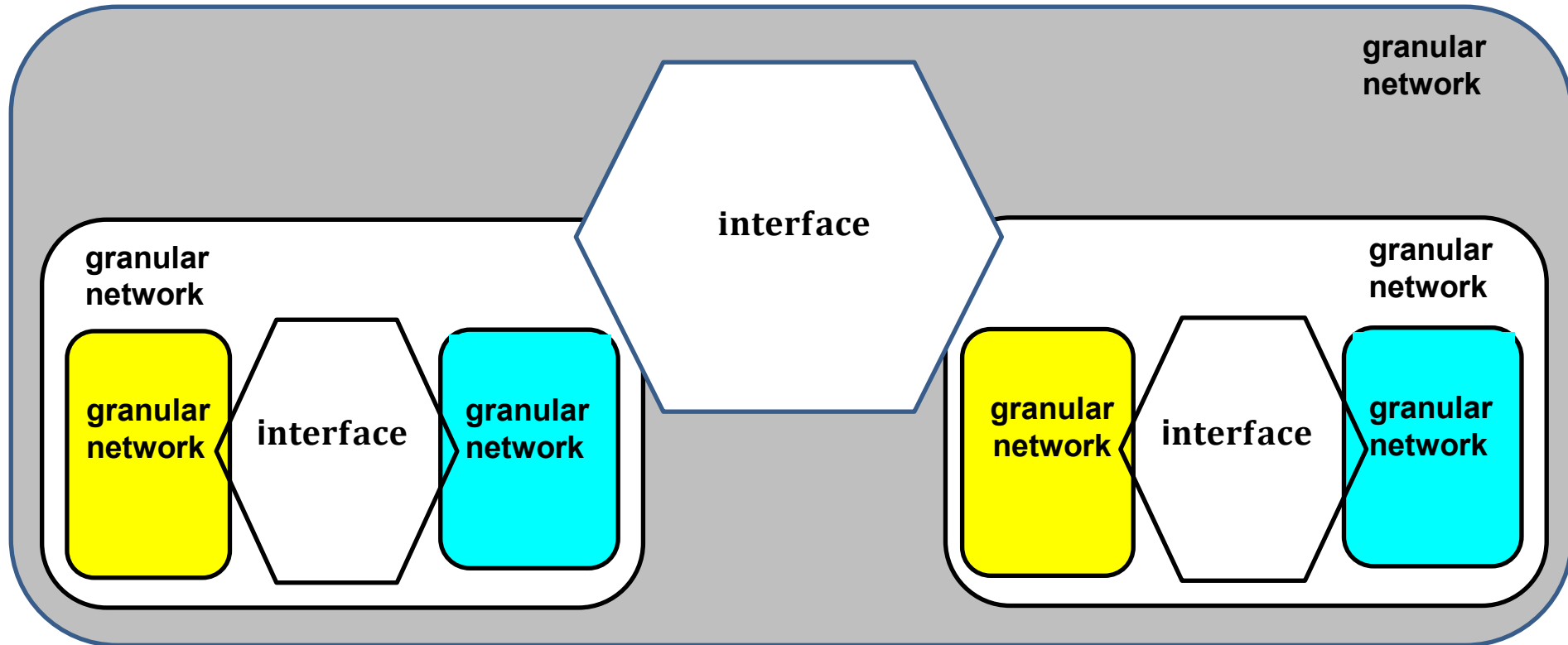
GN_2 granular network

- elementary (atomic) granules
- transformations for constructing granules
- methods (algorithms) for generation new granules, modifying existing ones and reasoning about computations over them
- can have nested structure
- ...

LINKING GRANULAR NETWORKS BY INTERFACES

PRODUCTS WITH CONSTRAINTS OF ALREADY CONSTRUCTED GRANULAR NETWORKS

granular network can have nested structure:



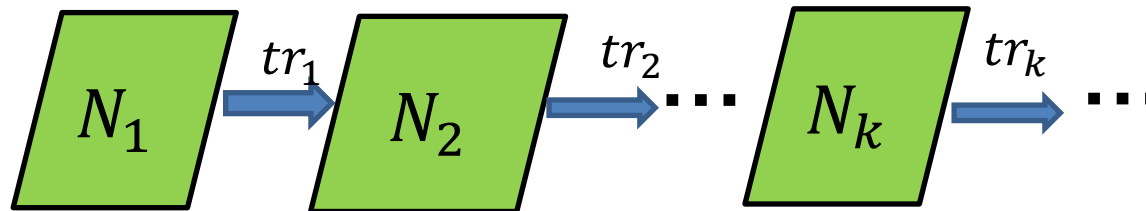
GRANULAR COMPUTATIONS IN GrC

GrC:

$N_1, N_2 \dots, N_k$ -- granular networks in the abstract space

$tr_1, tr_2, \dots, tr_k$ -- transformations realized in the abstract space

The control of c-granules, whether cooperating or competing with other c-granules, aiming to generate a granular computation along which is constructed an approximate solution of the problem to be solved to the required quality.



**MOTIVATIONS FOR
NEW COMPUTING MODEL
FOR
MODELING APPROXIMATION SPACES
IN ISs WITH BEHAVIOR
THAT CAN BE DIFFICULT TO SEPARATE
FROM INTERACTIONS
WITH COMPLEX REAL-WORLD
PHENOMENA AND/OR
COMPLEX ADAPTIVE SYSTEMS**

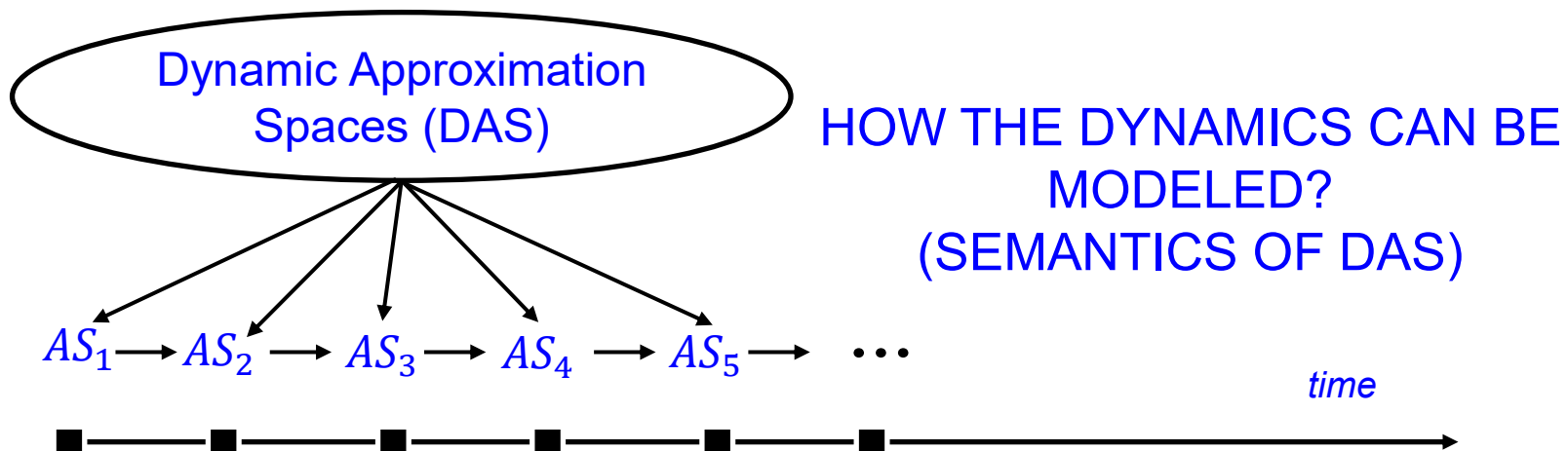
APPROXIMATION OF GRANULES (e.g., CONCEPTS /CLASSIFICATIONS) BASED ON ROUGH SETS

- Approximation of crisp granules (e.g., concepts/classifications) in crisp universes (e.g. approximation regions)
- Approximation of crisp granules (e.g. concepts/classifications) in crisp universes under uncertainty about granules and universes (e.g., rule-based classifiers)
- Dynamic Approximation Spaces are necessary
 - Approximation of vague granules (e.g., concepts/classifications) perceived by ISs.

Boundary regions based on the Pawlak model are crisp but
Boundary regions of vague concepts are not crisp!

Keefe, R. (2000) Theories of Vagueness. Cambridge Studies in Philosophy, Cambridge, UK)

- Separating evolving approximation spaces (e.g., for concepts or classifications) from interactions with complex real-world phenomena and/or complex adaptive systems involved in IS behavior can be difficult.



APPROXIMATION SPACES (ASs) IN ISs

- ASs are not given a priori, but rather, they should be discovered by ISs.
- ASs are dynamic objects.
- Dynamic Approximation Spaces (DAS) - postulates:
 - The dynamics of AS are grounded in the relevant computing model.
 - Making the right decisions in real-world situations requires synchronizing language, reasoning, action, and perception issues.
 - The adaptation of the current AS is realized through the IS, which includes this AS using the perceived properties of abstract and physical objects in the real world, as well as their interactions.
 - DAS modeling is considered in the context of ISs that interact with difficult-to-separate elements of systems of systems (SoS), which are composed of complex real-world phenomena and/or complex adaptive systems.
 - Developing adaptation strategies for ASs also requires synchronizing language, reasoning, action, and perception issues.
 - ...

Conclusion:

A relevant computing model on which DAS can be based is needed.

THE RELEVANT COMPUTING MODEL: FOUNDATIONS FOR DESIGN AND ANALYSIS OF INTELLIGENT SYSTEMS DEALING WITH COMPLEX REAL-WORLD PHENOMENA (IS's)

Many partial proposals in many different domains exist, e.g., complex (adaptive) systems, multi-agent systems, embodied AI, machine learning, robotics, cognitive science, neuroscience, computational intelligence, web intelligence, granular computing, natural computing, multisensory learning, cyber-physical systems, Internet of things.

Still we need

**the relevant computing model for developing the
foundations of IS's.**

**WE PROPOSE IGrC AS A MODEL THAT PROVIDES THE
BASIS FOR DEVELOPING INSIGHTFUL REASONING
METHODS FOR UNDERSTANDING SITUATIONS IN THE
REAL PHYSICAL WORLD TO A SATISFACTORY DEGREE,
TO MAKE THE RIGHT DECISIONS.**

NEEDS FOR RELEVANT MODEL OF COMPUTATIONS:

EXAMPLE OF CONCEPT OF MULTI-AGENT SYSTEM

An important open issue in the Agent Based Simulation field is **the lack of an univocal definition of the term “agent”** and of the paradigms and methodology used to build models; in software-engineering, a system of independent programs is considered a multi-agent system, **although there is no clarity about what the term exactly defines**. The term “agent”, deriving from the Latin “agens”, identifies someone (or something) who acts; the same word can also be used to define a mean through which some action is made or caused. **The term is used in many different fields and disciplines, such as economics, physics, natural sciences, sociology and many others.**

Marco Remondino: Reactive and deliberative agents applied to simulation of socio-economical and biological systems. International Journal of Simulation 6(12-13) 11-25 (2025)

J. Fang et al.: A Comprehensive Survey of Self-Evolving AI Agents: A New Paradigm Bridging Foundation Models and Lifelong Agentic Systems (2025). doi: 10.48550/arXiv.2508.07407

DEALING WITH COMPLEX PHENOMENA

Mathematics and the physical sciences made great strides for three centuries by constructing simplified models of complex phenomena, deriving, properties from the models, and verifying those properties experimentally.

This worked because the complexities ignored in the models were not the essential properties of the phenomena. **It does not work when the complexities are the essence.**

Frederick Brooks: The Mythical Man-Month: Essays on Software Engineering. Addison-Wesley, Boston, 1975. (extended Anniversary Edition in 1995).

Frederick Brooks introduced the distinction between:

Essential complexity — inherent to the problem's nature; irreducible.

Accidental complexity — due to imperfect tools; reducible.



BEYOND THE TURING TEST & REASONING

The Turing test, as originally conceived, focused on language and reasoning; **problems of perception and action were conspicuously absent**. The proposed tests will provide an opportunity to bring four important areas of AI research

(language, reasoning, perception, and action)

back into sync after each has regrettably diverged into a fairly independent area of research.

C. L. Ortiz Jr. Why we need a physically embodied Turing test and what it might look like. AI Magazine 37 (2016) 55–62.

PHYSICAL SEMANTICS

Constructing the **physical part of the [learning] theory** and unifying it with the mathematical part should be considered as one of the main goals of statistical learning theory

*Vladimir Vapnik, Statistical Learning Theory, Wiley 1998,
(Epilogue: Inference from sparse data, p. 721)*

WHAT IS A COMPUTATION ?

Two main problems of Computer Science:

What is a state?
What is a transition relation?

*What's an algorithm?
Yuri Gurevich (2011)*

<https://www.youtube.com/watch?v=FX2J24u92GI>

WHAT IS A COMPUTATION ?

It seems that we have no choice but to recognize the **dependence of our mathematical knowledge (...) on physics**, and that being so, it is time to abandon the classical view of computation as a purely logical notion independent of that of computation as a physical process

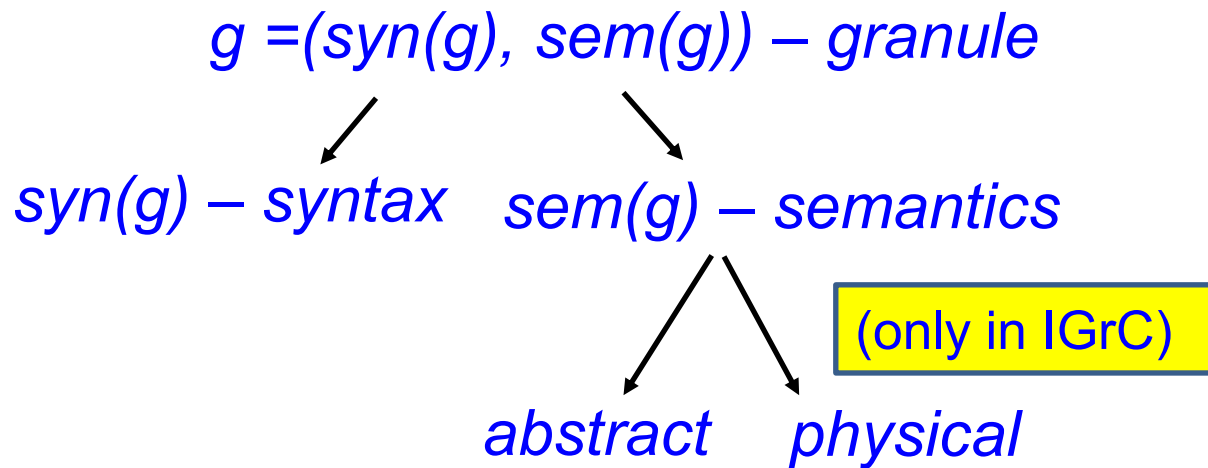
*David Deutsch, Artur Ekert, and Rossella Lupacchini,
Machines, logic and quantum physics.
Bull. Symbolic Logic 6 (2000) 265–283, p. 268*

PRELIMINARIES OF IGrC COMPLEX GRANULES (C-GRANULES)

**GRANULAR COMPUTING (GrC):
GRANULES CLOSED
IN THE ABSTRACT SPACE ONLY**

**INTERACTIVE GRANULAR
COMPUTING (IGrC):
GRANULES
IN THE ABSTRACT AND PHYSICAL SPACES
FOR PERCEPTION MODELING**

GRANULAR COMPUTING (GrC) & INTERACTIVE GRANULAR COMPUTING (IGrC)



FROM GrC TO IGrC

GrC, with abstract information granules as basic objects, is generalized to IGrC through the introduction of complex granules (c-granules), which are the fundamental objects of IGrC. These combine abstract and physical objects, enabling the perception of their properties through the control of c-granules. The IGrC model differs from the classical Turing model. In the IGrC model, both language and reasoning issues as well as actions and perception are significant. Research on IGrC utilizes existing partial results from various fields, such as multi-agent systems, perception and action, machine learning, natural language processing, and more.

INFORMATION GRANULES OF GrC AS SPECIAL CASES OF C-GRANULES OF IGrC

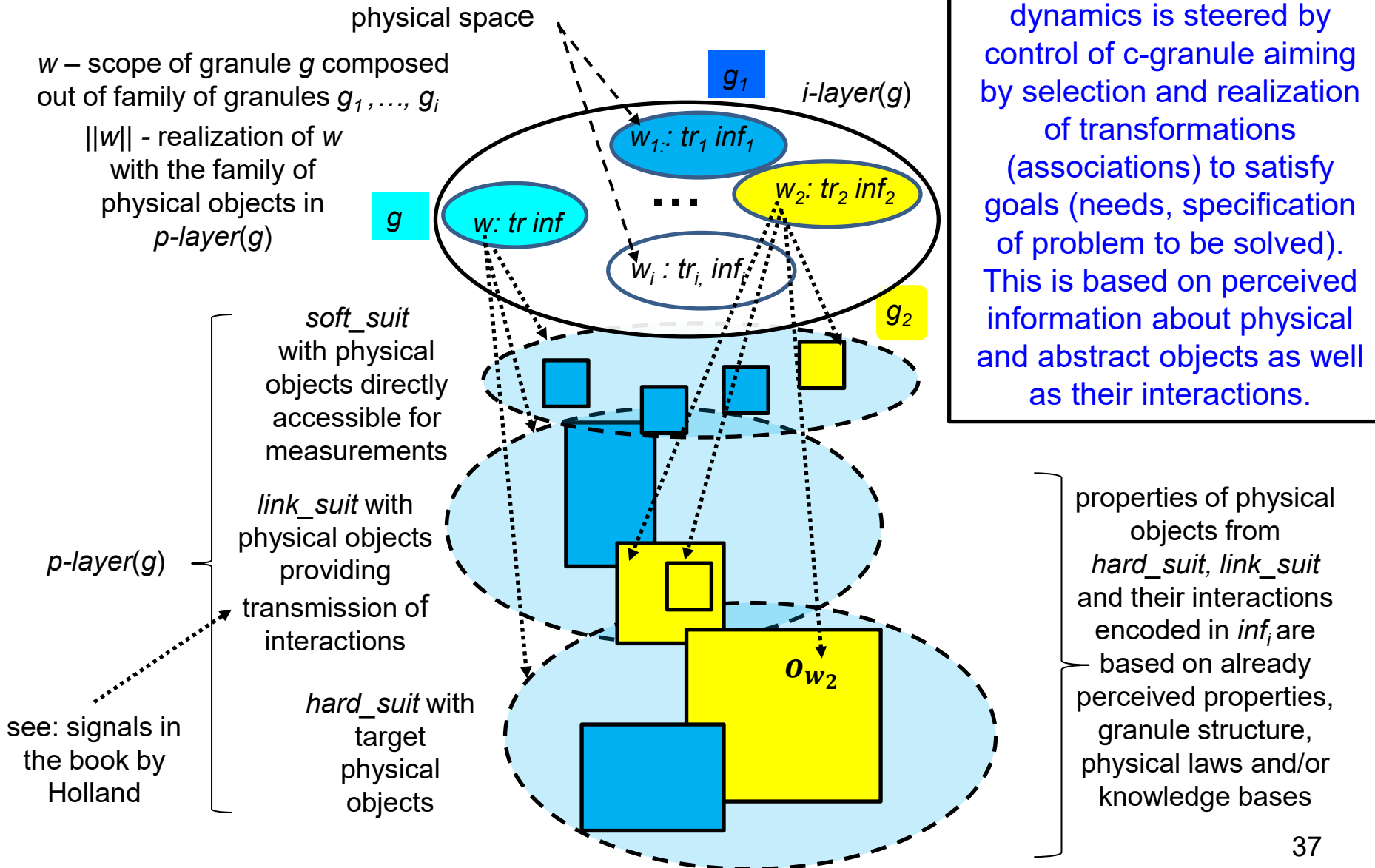
The IGrC model substantially extends the GrC model. GrC granules can be considered special cases of c-granules of IGrC. Any information granule g_o from GrC can be identified as a c-granule g consisting of an i-layer that encodes g_o and two transformation specifications: *store* and *read* (labeled by the address that points to physical memory). The control of the c-granule g can use these specifications to either store g_o in the p-layer of g or read g_o from it into the i-layer of g (without losing any information).

C-GRANULE: INTUITION

specifications of (families of) spatio-temporal windows realized by control as physical pointers to the corresponding parts of the physical space

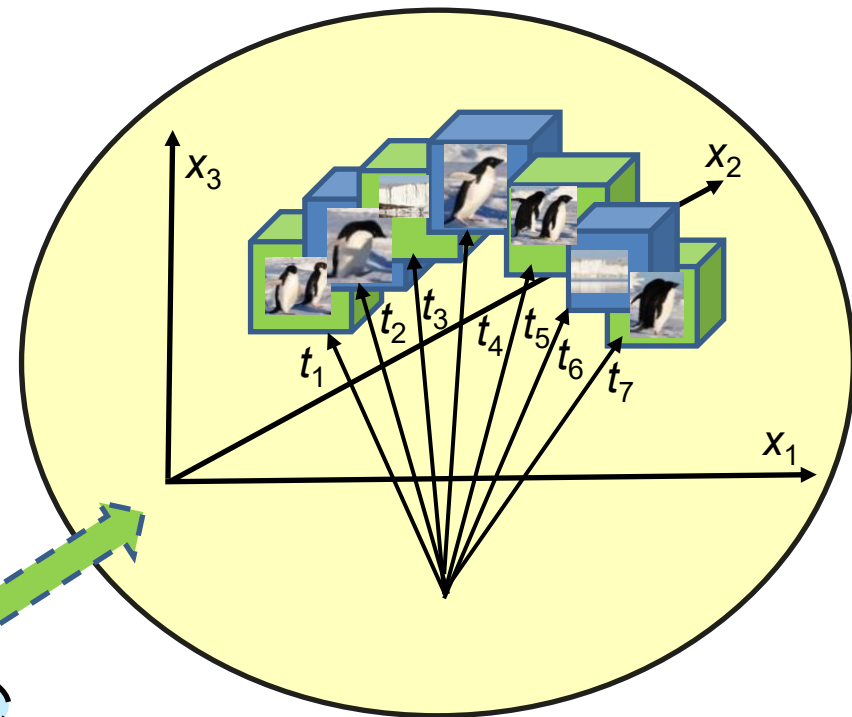
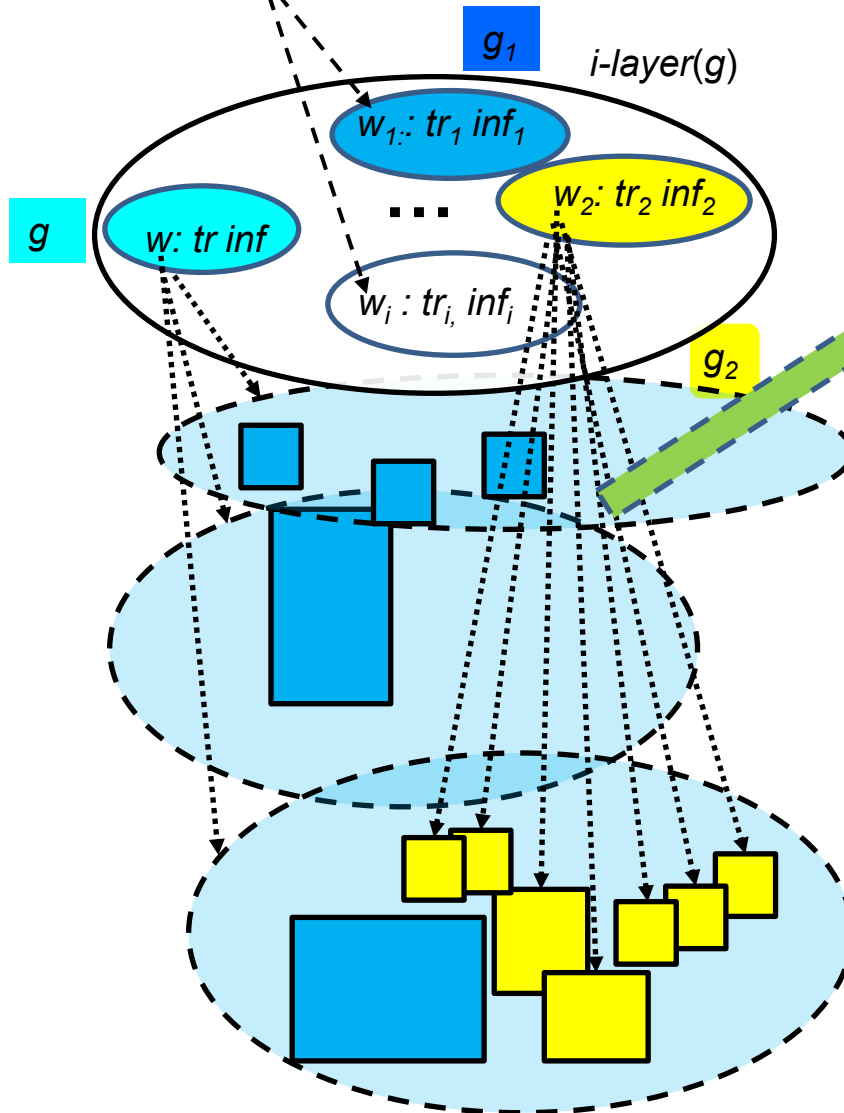
w – scope of granule g composed out of family of granules $g_1, \dots, g_i$

$||w||$ - realization of w with the family of physical objects in $p\text{-layer}(g)$



C-GRANULE: INTUITION

specifications of (families of) spatio-temporal windows realized by control as physical pointers to the corresponding parts of the physical space



inf_2 includes video information gathered during physical semantics realization of specification of transformation tr_2

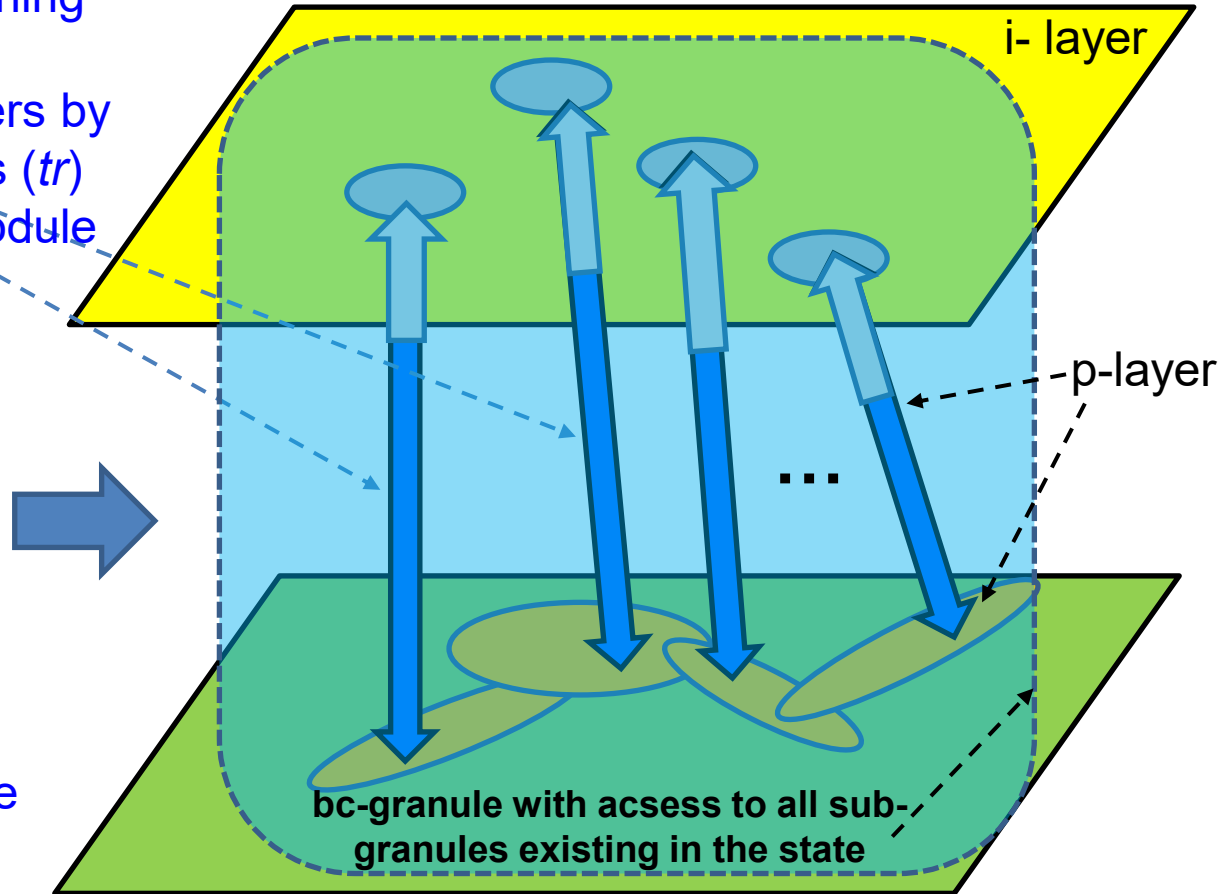
(e.g. tracking objects what may require cooperation with cameras placed in *soft_suit* or *link_suit*) from parts of the physical space pointed by spatio-temporal window specification w_2 ; inf_2 also includes information about the expected results of tr_2 realization; these results may be different from the actual results due to interactions with the environment

C-GRANULE WITH CONTROL: INTUITION

Control is involved in establishing associations between the informational and physical layers by implementing transformations (tr) through its implementation module (IM) (physical semantics).

CONTROL
of c-granule as a
subgranule of c-
granule

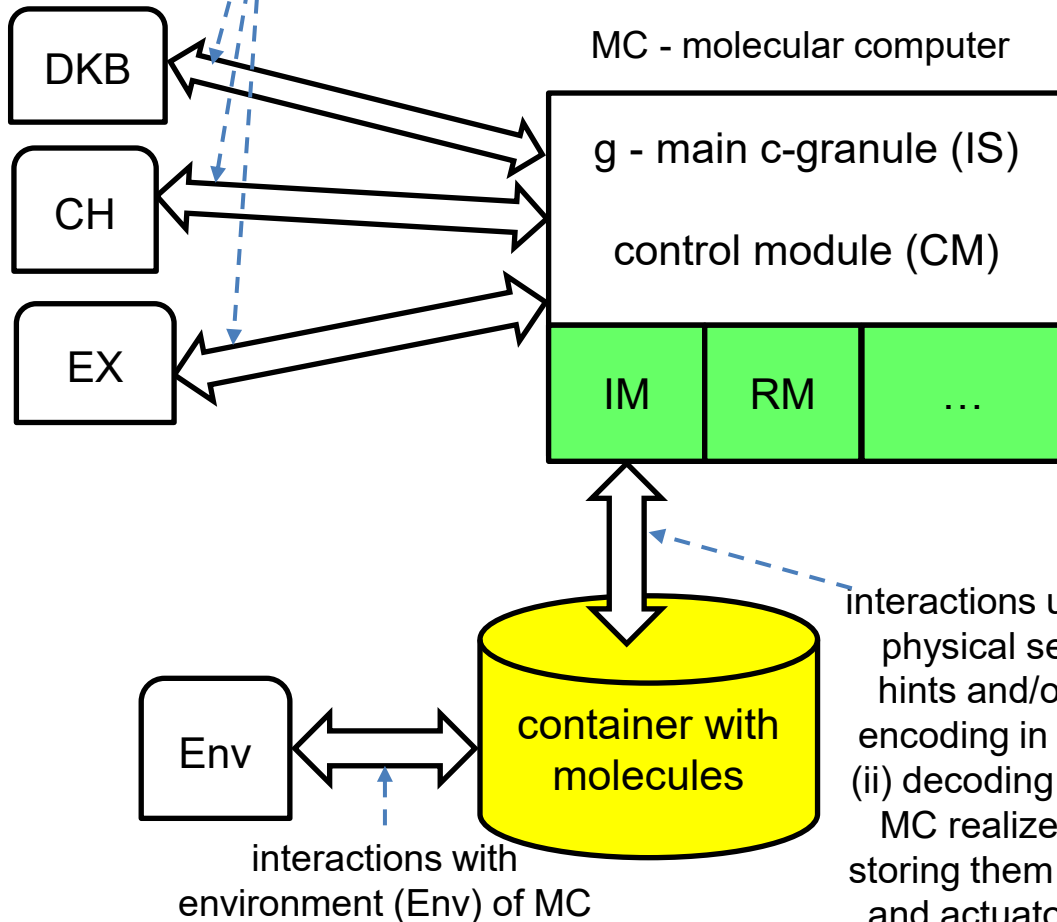
Control is initiating communications between the informational layer and the physical layer using relevant c-granules (generated by its IM), allowing the collection of properties of perceived physical objects and their interactions within the informational layer.



SIMPLIFIED VIEW OF NETWORK OF SUB-GRANULES INTERACTING WITH ABSTRACT AND PHYSICAL OBJECTS GENERATED AND STEERED UNDER SUPERVISION OF A GLOBAL SUBGRANULE OF THE CONTROL RESPONSIBLE FOR BASIC CYCLE (bc-granule)

ILLUSTRATIVE EXAMPLE OF COMPUND C-GRANULE: MOLECULAR COMPUTER (MC)

interactions realized by dialogues resulting in presenting to the control module (CM) of g the problem specification and domain knowledge/physical laws as well relevant hints concerning the realization of physical semantics of the specification in molecular computer (MC)



structure of g :

sub-granules of g :
CM - control module

...

sub-granules of CM:

IM - implementational module
RM - reasoning module

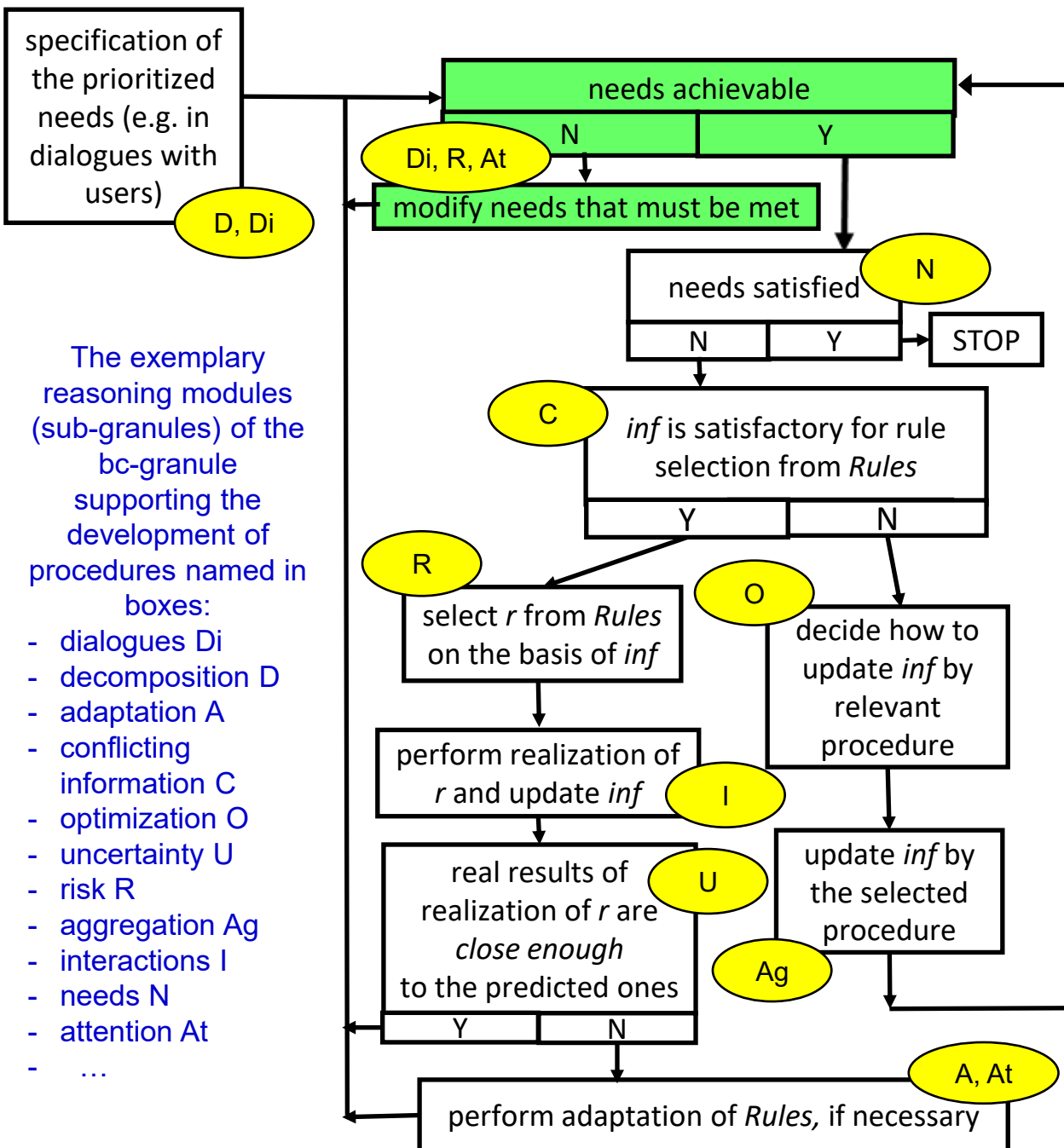
...

sub-granules of g realizing interactions with: domain knowledge bases (DKB), experts (EX), chatbots (CH)...
MC

...

interactions used to realize in MC the provided by experts physical semantics of the problem specification using hints and/or domain knowledge provided by CM for (i) encoding in the MC of the input of problem to be solved, (ii) decoding the results of highly parallel computations of MC realized by molecules and their aggregations and storing them in informational layer of g , (iii) using sensors and actuators, (iv) steering of interactions of molecules and their aggregations in MC and of the container content,

REASONING SUPPORTING THE BASIC CONTROL CYCLE OF BC-GRANULE



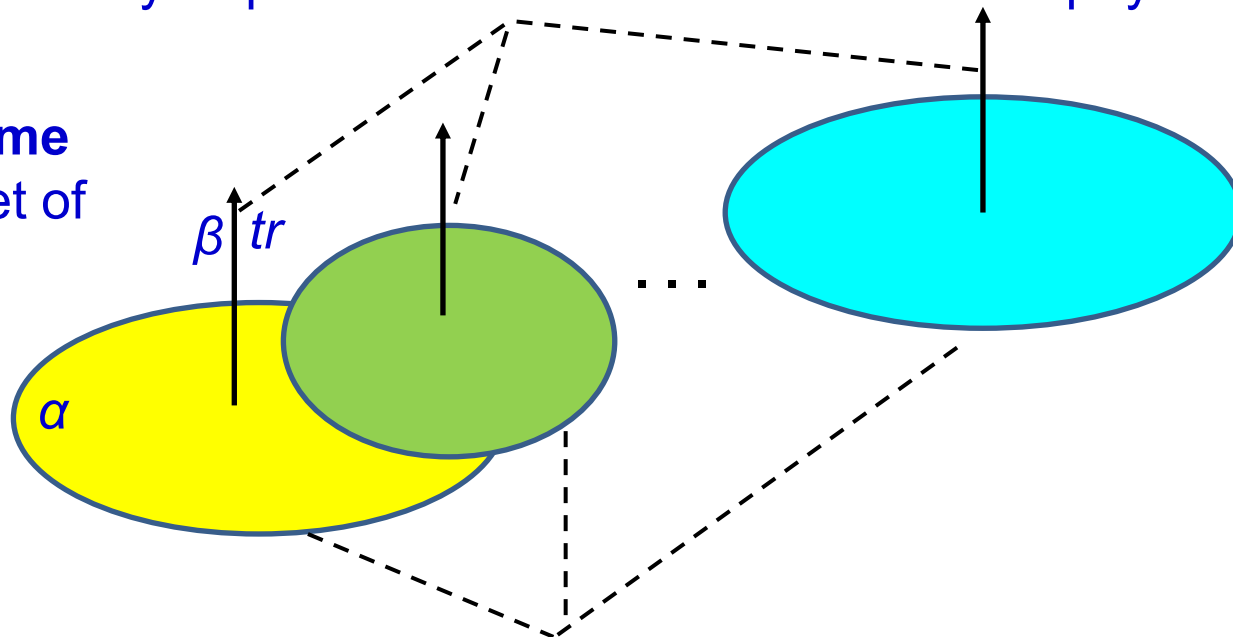
Many reasoning modules (sub-granules) are supporting realization of the basic cycle by bc-granule. Advanced reasoning may support two different forms of control behavior in the realization of the basic cycle (bc-granule): *reactive* and *deliberative*. For example, the control of bc-granule may recognize that there are some arguments supporting the satisfiability of a concept that signals a high risk of reaching a very dangerous state if an immediate reaction is not taken. In other situations, there is more room for deliberation, which aims to better understand the current situation by taking additional measurements, performing actions, extracting information from knowledge bases or performing adaptation (e.g., attention). One may recognize an analogy here to Kahneman's *fast* and *slow* reasoning.

inf – information in i-layer about the currently perceived situation
Rules – set of rules for transformation of sub-granules

DISCOVERY AND EVOLUTION OF COMPLEX GAMES INTERACTING WITH THE ABSTRACT AND PHYSICAL WORLDS

transformations (associations) (tr), with specifications concerning their expected results (β), aiming to perform the relevant measurements/ actions/ plans toward achieving the expected target goals; the implementation of transformations may require interactions with abstract and physical objects

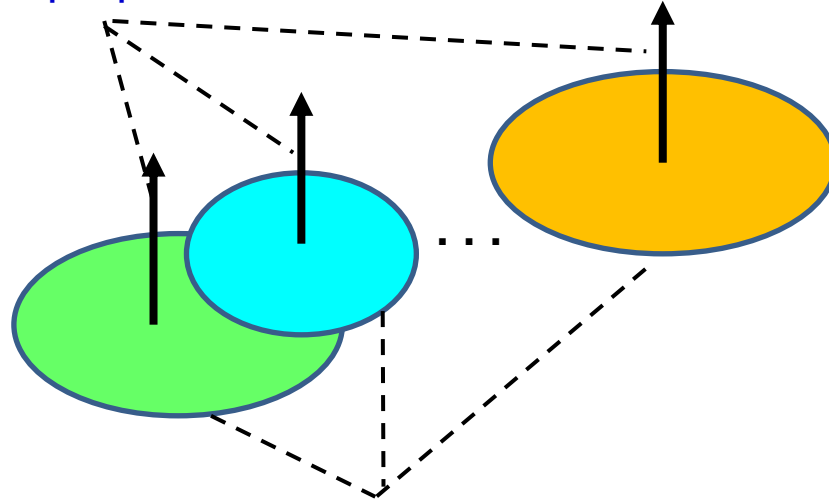
complex game
i.e., a finite set of
rules
 $r: \alpha \rightarrow_{tr} \beta$



complex vague concepts (α) triggering actions/plans;
their approximations by classifiers may require interactions with abstract and
physical objects

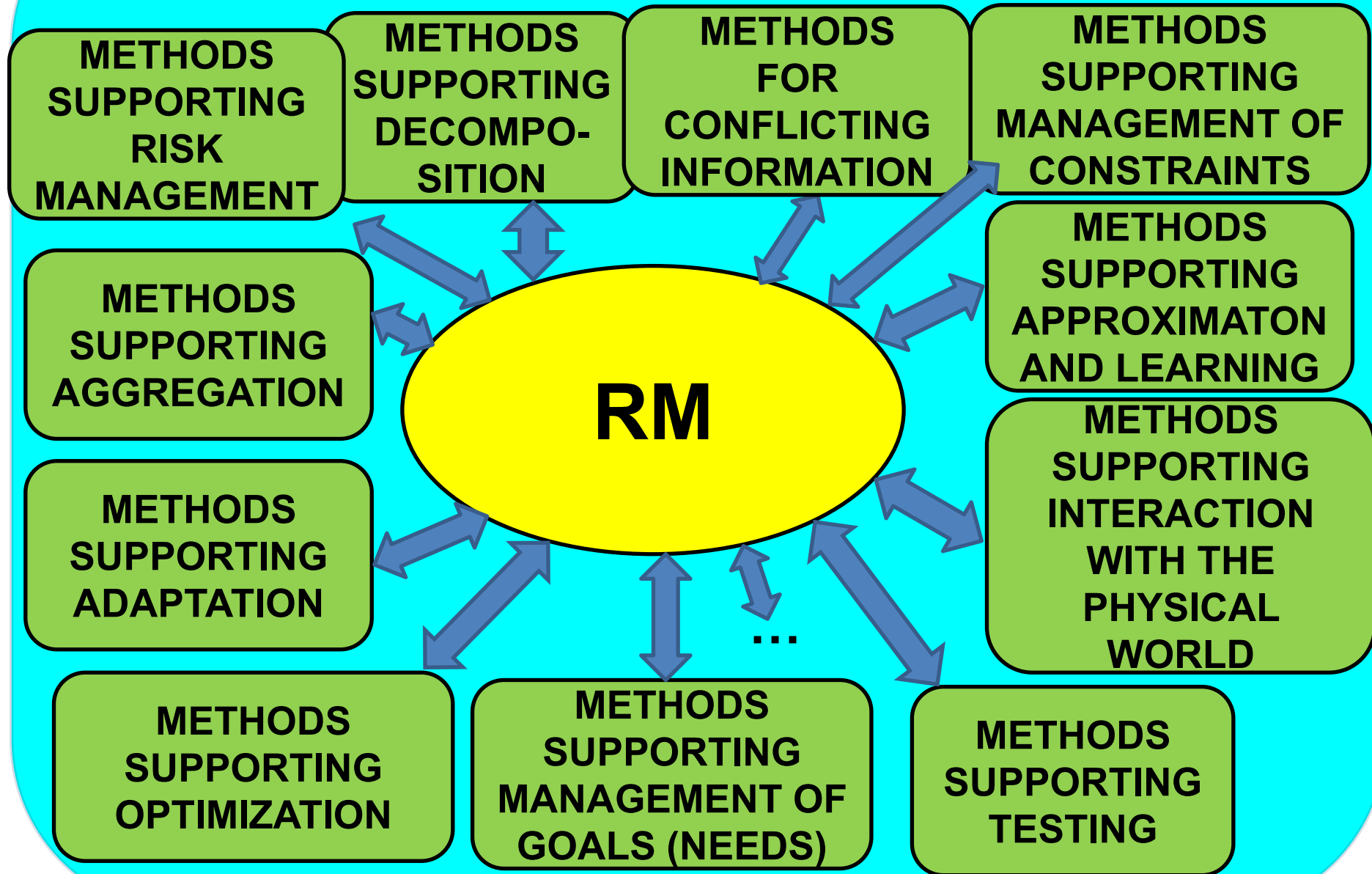
DISCOVERY OF COMPLEX ADAPTIVE GAMES INTERACTING WITH THE ABSTRACT AND PHYSICAL WORLDS

complex games for situations with the relevant
properties

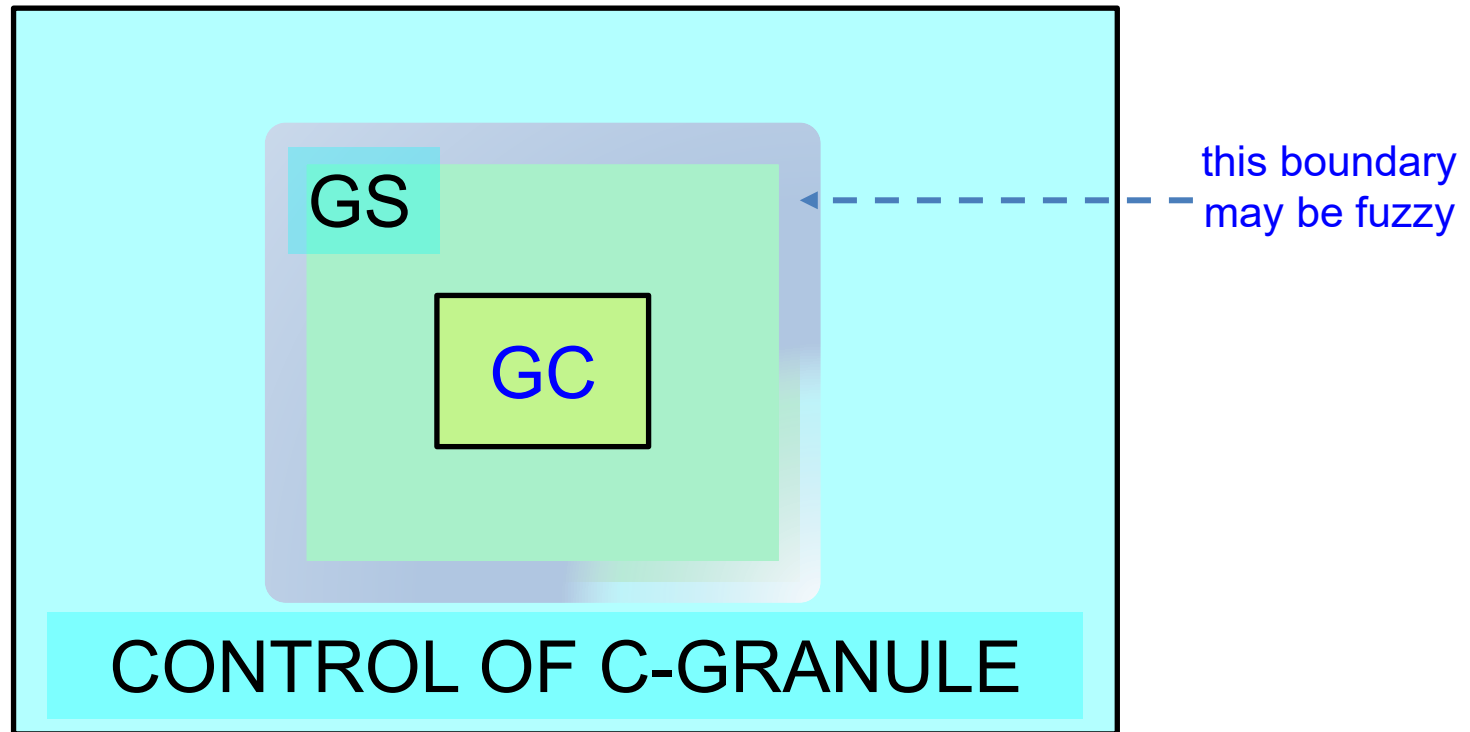


complex vague concepts triggering complex
games

INTERRELATED SUB-GRANULES OF REASONING MODULE (RM) SUPPORTING BEHAVIOR OF CONTROL



GRANULAR SPACE: CONTEXT OF GRANULAR CALCULUS



Remark. In the case of societies of c-granules, the problem of steering their granular computations becomes more complex. Proper cooperation and competition mechanisms between c-granules (including their coalitions) within and between societies become important. The goal is to ensure that societies behave in a satisfactory manner when solving problems, i.e., to produce high-quality approximate solutions encoded in generated granular computations.

GRANULAR NETWORKS AND THEIR COMPONENTS (LIKE GRANULAR SPACES OR INTERFACES) AND GRANULAR COMPUTATIONS OVER THEM OR EVEN SETS OF COMPUTATIONS (DEFINED BY THEIR PROPERTIES) CAN BE TREATED AS MORE COMPOUND C-GRANULES. THE SEMANTICS OF THEM IS DEFINED IN THE CONTEXT OF THE CONTROL OF C-GRANULE (e.g., IS) AND INTERACTIONS WITH THE PHYSICAL WORLD INCLUDING COMPLEX PHENOMENA IN THE REAL WORLD AND OTHER C-GRANULES.

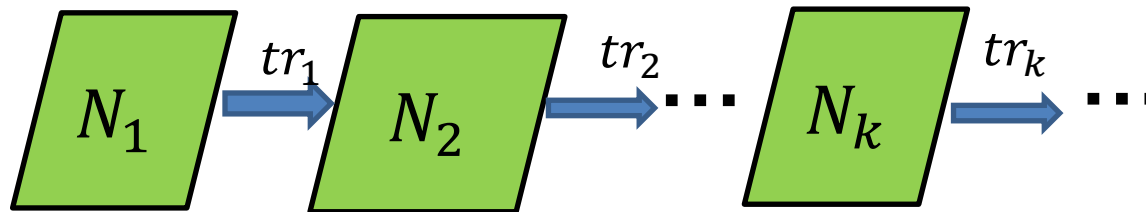
GRANULAR COMPUTATIONS IN GrC and IGrC

GrC:

$N_1, N_2 \dots, N_k$ -- granular networks in the abstract space

$tr_1, tr_2, \dots, tr_k$ -- transformations realized in the abstract space

The control of c-granules, whether cooperating or competing with other c-granules, aiming to generate a granular computation along which is constructed an approximate solution of the problem to be solved to the required quality.



IGrC:

$N_1, N_2 \dots, N_k$ -- granular networks in the abstract and physical spaces

$tr_1, tr_2, \dots, tr_k$ -- transformations realized in the abstract and physical spaces

The control of c-granules, whether cooperating or competing with other c-granules, engages in interaction with the physical space, aiming to generate a granular computation along which is constructed an approximate solution of the problem to be solved to the required quality.

ROUGH SETS (RS) IN IGrC

FROM STATIC TO DYNAMIC APPROXIMATION SPACES

EVOLUTION OF APPROXIMATION SPACES IN THE ROUGH SET APPROACH

APPROXIMATION SPACE IN THE PAWLAK ROUGH SET MODEL

$$AS = (U, r)$$

...

APPROXIMATION SPACE IN THE
MULTIRELATIONAL OR MULTIGRANULATION ROUGH SET MODEL

...

APPROXIMATION SPACES IN IGrC ARE EVOLVING
IN DYNAMIC GRANULAR NETWORKS
INTERACTING WITH COMPLEX REAL-WORLD PHENOMENA
AND/OR COMPLEX ADAPTIVE SYSTEMS
BEING PARTS OF SYSTEMS OF SYSTEMS (SoS)

The discovery of dynamic granular networks through the control of c-granules (interacting with the environment) is the foundation for finding relevant computational building blocks [using terminology that Leslie Valiant would use] for adaptive approximations of concepts or classifications and/or approximate problem solving.

evolution

ROUGH SETS (RS) & IGrC

- Evolution of approximation spaces (AS) since 1982.
- In IGrC, approximation spaces are special sub-granules of controlling granules (i.e., c-granules that model the control of c-granules involved in granular computations).
- The construction and adaptation of AS in IGrC should be considered in the context of the controlling granules that include them. This means the controlling granules perform construction and adaptation of AS (while interacting with the abstract and physical worlds).
- Challenge for foundations of AI systems: discovery of approximate solutions of problems.

APPROXIMATION SPACES (COMPOUND C-GRANULES) AS PARTS OF COMPLEX ADAPTIVE SYSTEMS

Similarly to how natural language is treated as an adaptive complex system, models of approximation spaces should have the following properties:

- They consist of multiple c-granules (societies of c-granules) that interact with one another.
- They are adaptive, meaning c-granule behavior is based on past interactions. Current and past interactions together influence future c-granule behavior.
- The behavior of c-granules is the consequence of competing factors ranging from perceptual constraints to social motivations.

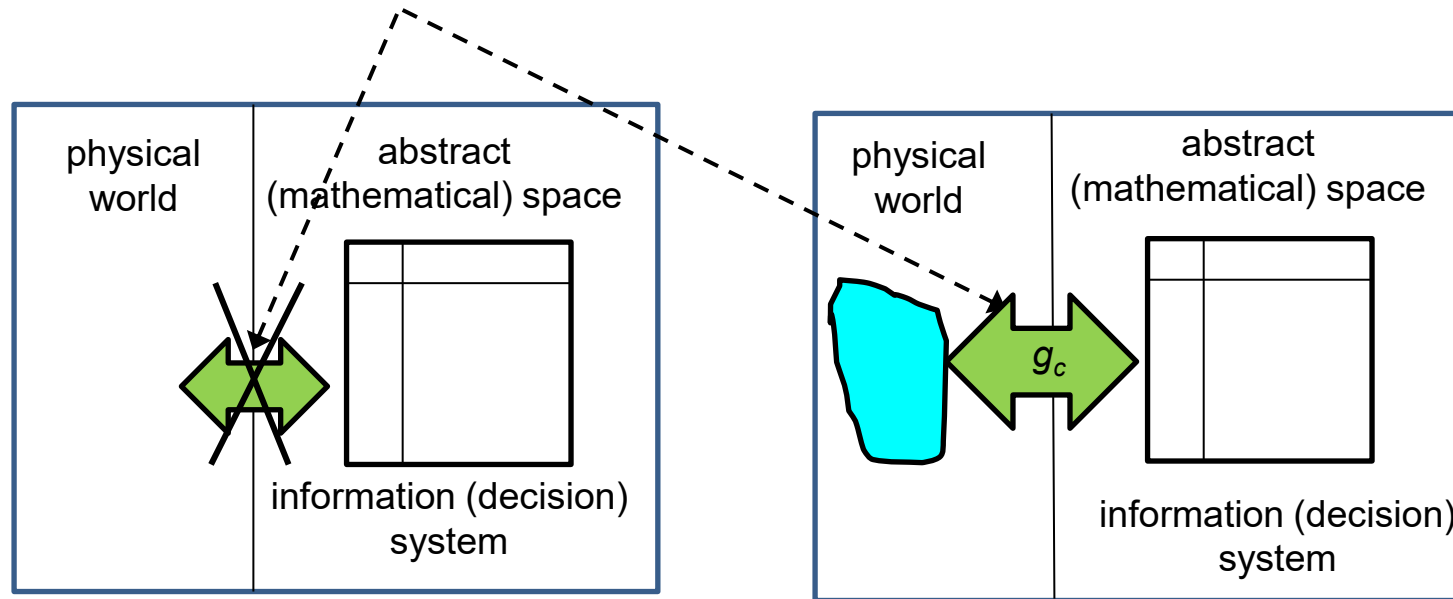
*N. C. Ellis, D. Larsen-Freeman: Title: Language as a Complex Adaptive System.
Wiley-Blackwell (2009)*

This has important implications for modeling approximation spaces. Specifically, such modeling cannot be based solely on a computing model that is closed in an abstract space, such as the Turing model. The granular computations involved in modeling approximation spaces depend on physical laws and are based on the interactions of abstract and physical objects. We propose using IGrC as the relevant computing model.

**APPROXIMATION SPACES IN IGrC
ARE NOT ISOLATED OBJECTS, i.e.
OBJECTS THAT CAN BE SEPARATED
FROM INTERACTIONS WITH THE
ENVIRONMENT**

EXAMPLE: INFORMATION(DECISION) SYSTEMS ARE PARTS OF DYNAMIC OBJECTS: C-GRANULES

Continuous interactions with the physical world during perceiving of the current situation aiming to understand this situation to a degree satisfactory for making the rights decisions



In the existing approaches to rough sets interactions with the physical world are omitted. Information systems are **GIVEN** as pure mathematical objects.

Rough sets in IGrC (perceptual approach) based on **physical semantics**:: information (decision) systems are obtained as the result of granulation of information perceived by c-granule g_c in the physical world.

APPROXIMATION SPACES IN IGrC APPEAR IN THE CONTEXT OF OTHER COMPLEX OBJECTS AND INTERACT WITH THEM

AMONG SUCH OBJECTS ARE

- **CONTROL OF ISs (main c-granule) AND ITS SUB-GRANULES (MODULES)**
- **GRANULAR COMPUTATIONS OF ISs**
- **COMPLEX REAL-WORLD PHENOMENA AND/OR COMPLEX ADAPTIVE SYSTEMS INTERACTING WITH ISs**

APPROXIMATION SPACES IN IGrC

Temporary approximation spaces consist of c-granules (granules, for short), methods for generating granules, and their relevant transformations for understanding the perceived situation by the c-granule control. These spaces are represented in granular networks, on which c-granule control computations are performed. The control module (subgranule) of a given c-granule, interacting with the physical and abstract worlds steers its granular networks. Changes to temporary approximation spaces are represented in subsequent granular networks of granular computations. The objective of control is to generate the most relevant approximation spaces for each perceived situation.

Temporary approximation spaces provide granules that define different regions of approximated concepts, classifications, and sensor/actuator structures at various hierarchical levels. These granules are constructed through c-granule control and serve as computational building blocks for making the right decisions. These granules are used to create complex, adaptive, high-quality games that support c-granule control in guiding granular computations to satisfy c-granule needs.

APPROXIMATION SPACES IN IGrC

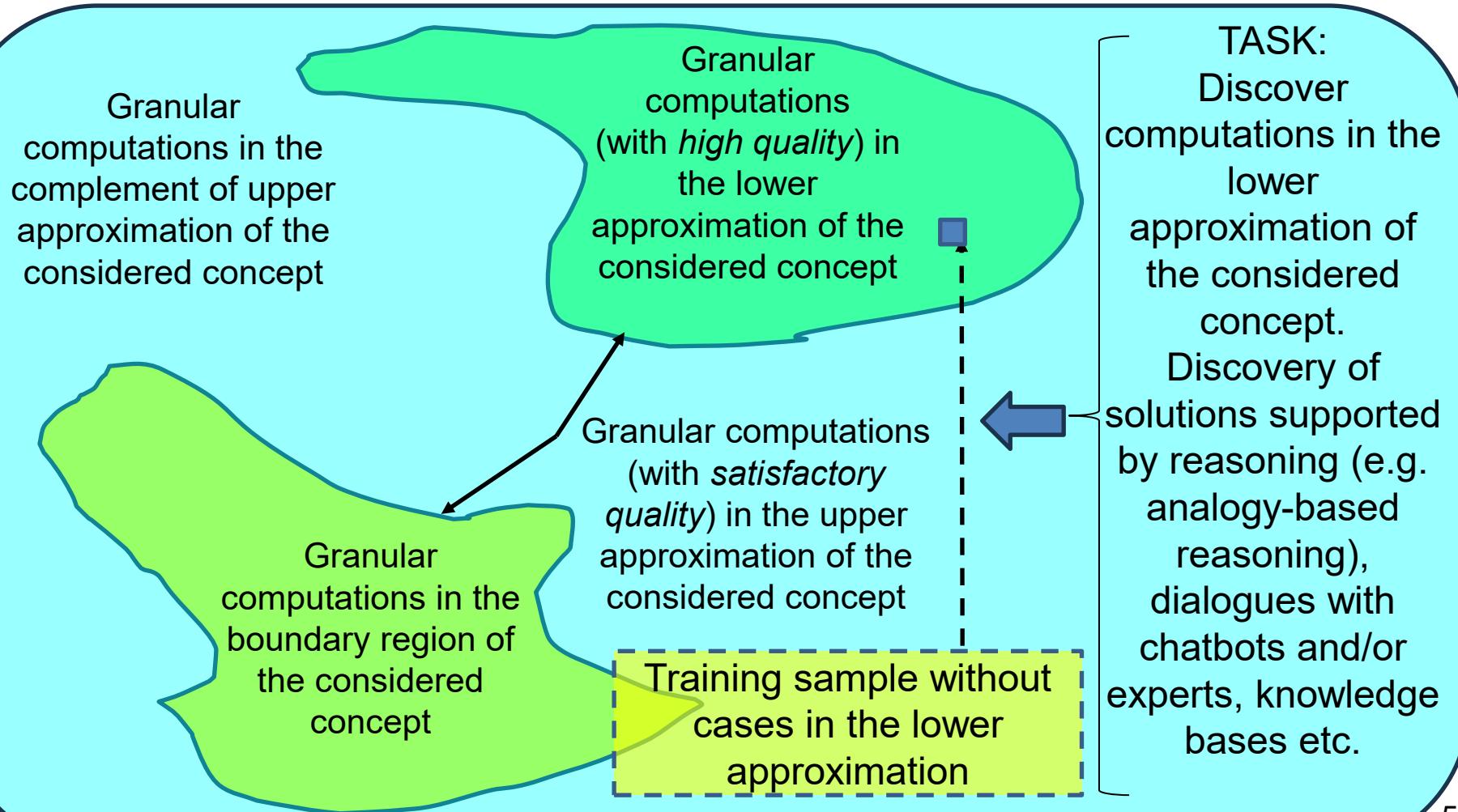
- In granular computations, approximation spaces of a given c-granule (or society of c-granules) are complex, dynamic structures.
- These approximation spaces do not have fixed a priori dynamics. Rather, they are partially determined by the control of the c-granule, which steers its interactions with the environment to meet its needs or goals.
- The states of c-granule computations are granular networks, particularly containing as sub-networks, networks of approximation spaces.
- A network of approximation spaces contains granular calculi that are linked by interfaces representing relationships between the calculi.
- C-granules from the granular calculi of a given network of approximation spaces are used as computational building blocks
 - for approximating complex, vague concepts used in constructing complex games,
 - over which reasoning is performed to support control of the c-granule behavior,
 - for constructing of strategies for adapting complex games.
- The successive states of a c-granule's granular computations contain complex games that are adaptations of the preceding ones.
- For important class of problems, the universe of objects for classification contains granular computations of c-granule and in this space are considered concepts to be approximated.

APPROXIMATION SPACES IN THE CONTEXT OF GRANULAR COMPUTATIONS OF ISs

APPROXIMATION AREAS OF THE CONCEPT:

GRANULAR COMPUTATIONS WITH HIGH-QUALITY SOLUTIONS OF THE SPECIFIED PROBLEM

IN THE DOMAIN CONSISTING OF GRANULAR COMPUTATIONS OF C-GRANULE CONTROL



RS – CURENT APPROACH

Basic concept:
Approximation Space (AS)
(static abstract object)



Basic task:
Approximation of concepts



RS & IGrC – FUTURE APPROACH

Basic concept:

The Dynamic Approximation Space (DAS) modeled by the IS (c-granule with control or society of c-granules) in the context of the CAS for the approximation of granules.

The interaction of granules with their granulation and de-granulation in IS supporting

- (i) the generation of rough set-based granular computations in interaction with the abstract and physical worlds and
- (ii) for a given problem specification to IS, DAS supports construction (in the form of c-granules) of "abstract and/or physical" approximate solutions of appropriate quality through properly steered granular computations over granular networks.



Basic task:

Discovery of approximate solutions to problems of appropriate quality based on discovery of rough set based complex game teams over dynamic rough set based granular networks forming granular computations.

EXAMPLARY DOMAINS WHERE DISCOVERING HIGH-QUALITY APPROXIMATE SOLUTIONS IS IMPORTANT

- Discovery of learning algorithms and construction of classifiers
- Automatic design of robots
- Drug discovery
- Algorithmic trading
- Evaluation of information provided by LMM (hallucinations)
- Generative AI
- Modeling cognitive computers
- Real-world applications in high-risk domains: autonomous driving, health care, and intelligent assistants

J. Qi, M. Li et al.: Towards trustworthy agentic AI: A comprehensive survey of safety, robustness, privacy, and system security. Academia AI and Applications 2(2) 1-36 (2026). doi: 10.20935/AcadAI8260

- Computers based on biological computing

Xu, J. (2025). Protein Computing. In: Biological Computing. Springer, Singapore (2025). doi: 10.1007/978-981-96-3870-3_12

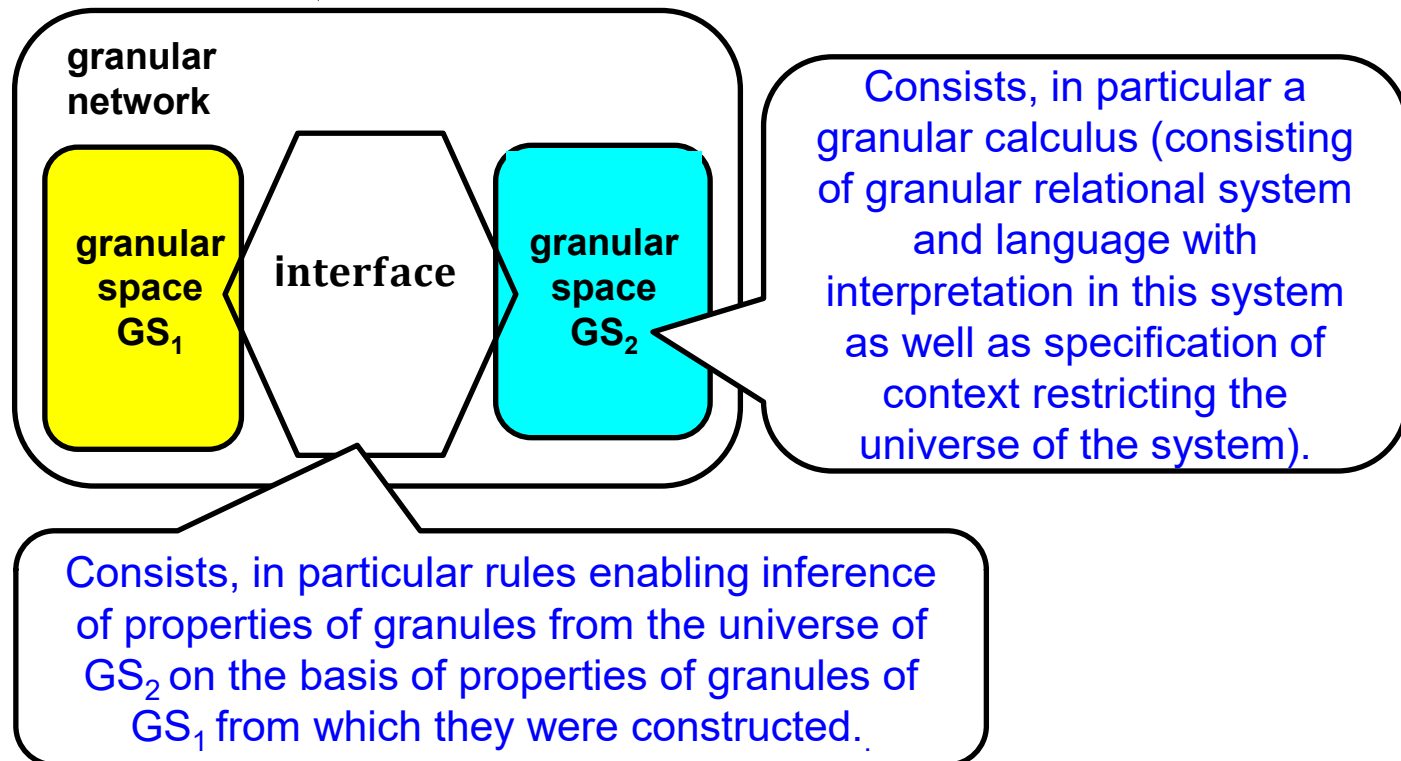
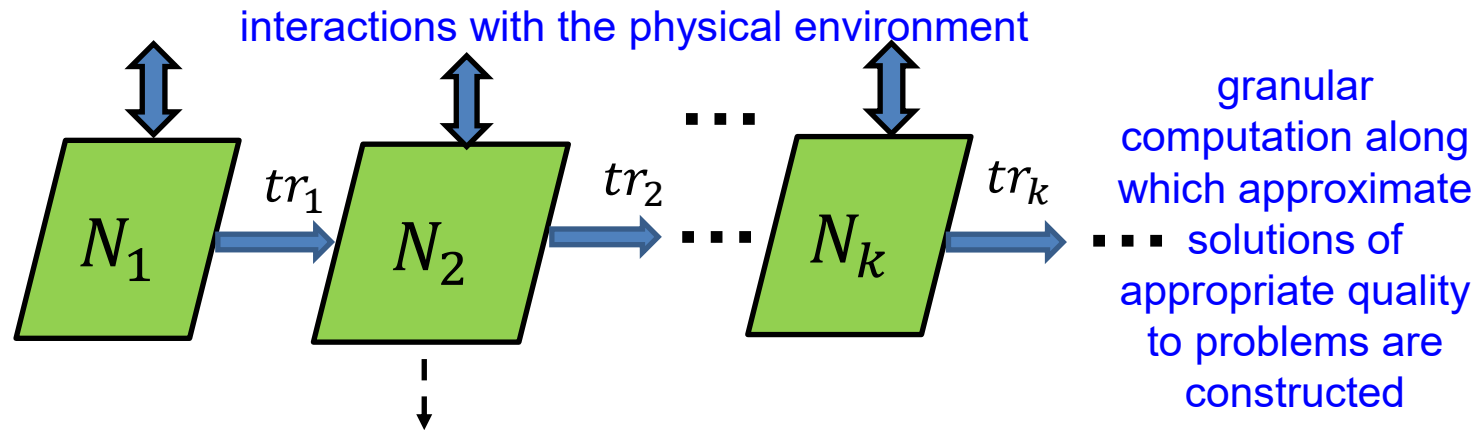
- ...

CHALLENGE: DEVELOPING THE FOUNDATIONS BASED ON IGrC AND RS FOR THE DESIGN AND ANALYSIS OF DISCOVERY AI SYSTEMS FOR DIFFERENT DOMAINS

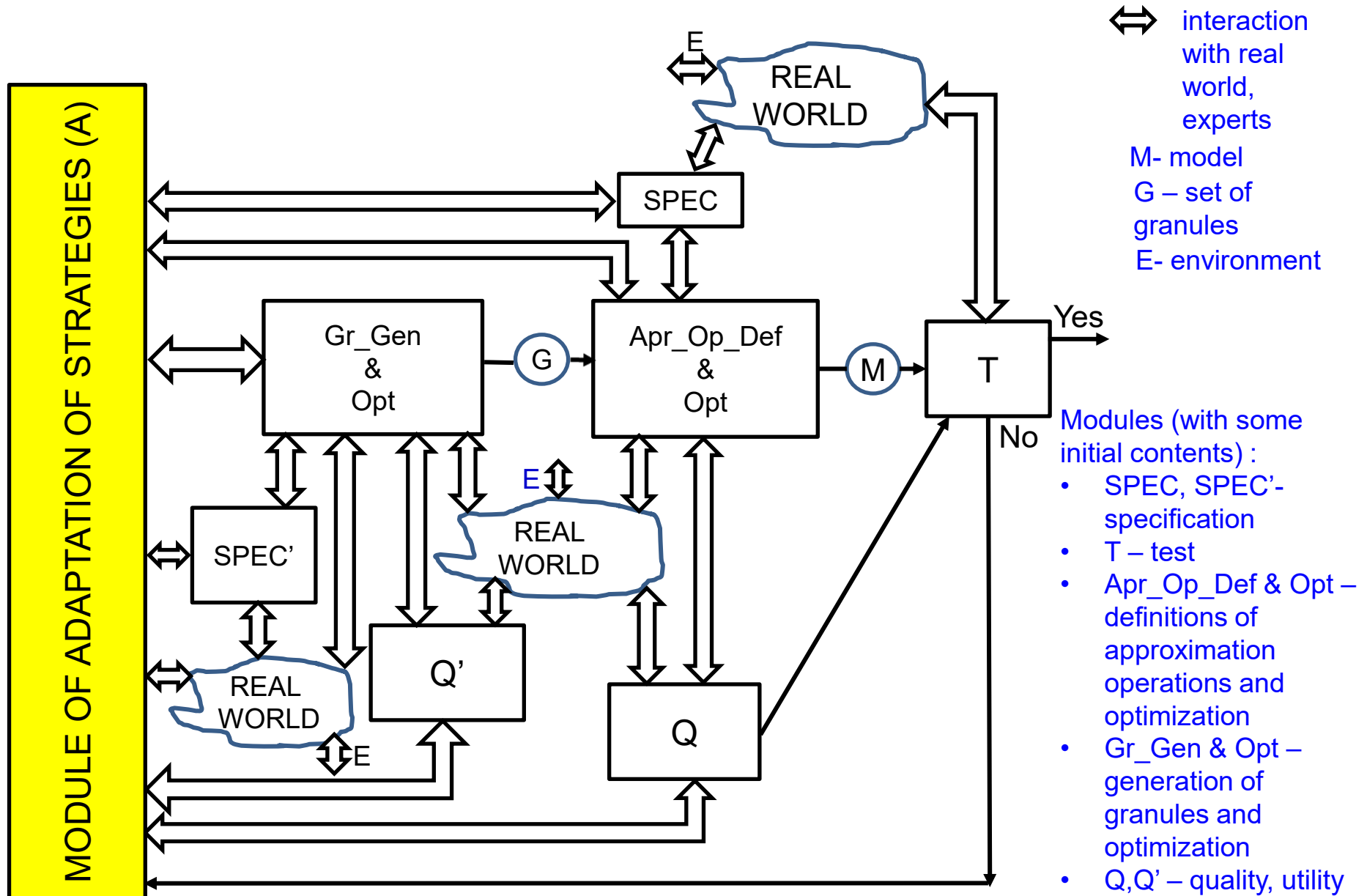
e.g., COMMON CORE OF SUCH SYSTEMS BASED ON IGrC

**ADAPTATION
OF
APPROXIMATION SPACES
SHOULD BE CONSIDERED IN THE
CONTEXT OF AFOREMENTIONED
COMPLEX OBJECTS**

ADAPTATION OF APPROXIMATION SPACES IN GRANULAR COMPUTATIONS



ADAPTATION SCHEME INTUITION FOR APPROXIMATION SPACES IN CONTROL OF C-GRANULE



DISCOVERY AS A PARADIGM OF LOGIC

When generating, e.g., a decision tree along granular computation (with granules being partitions), it is important to understand the reasoning behind **discovering** a heuristic for transforming the current partition. For example, the entropy (information) gain criterion suggests splitting an impure class in the current partition using the best attribute that has not yet been used to generate this class. The selection of the attribute is based on the difference between the impurity of the class being split and its subpartition created by that attribute.

Bocheński says that [...] *one can ask two different basic questions:*

(1) *What follows given premises?*

(2) *From what premises can a given sentence (conclusion) be deduced?*

Aristotle primarily considered the first question, **justification**, [...] but poses also the second, **discovery**, and tries to show **how the premises of a syllogism must be constructed in order to yield a given conclusion**

Bocheński, J. (1961). A history of formal logic. University of Notre Dame Press.

The logic of discovery is still in its infancy. Consequently, we do not yet have satisfactory automatic discovery methods supported by such logic. Consequently, AI systems cooperate with humans (see Human-Centered AI and Human-in-the-Loop Machine Learning) and chatbots in discovery tasks.

R. Hightower: Chapter 9: Leveraging Advanced Reasoning Models. In: Programming the OpenAI APIs with Python: A Comprehensive Guide to Building AI Applications. <https://rick-hightower.notion.site/Chapter-9-Leveraging-Advanced-Reasoning-Models-1e0d6bbdbbba808190d0e3fa0f26192c?pvs=21>

FOUNDATIONS BASED ON IGrC & RS FOR IS's DEALING WITH COMPLEX REAL-WORLD PHENOMENA

Tomorrow, I believe, we will use
[IS's]
to support our decisions
in defining our research strategy and specific aims,
in managing our experiments,
in collecting our results, interpreting our data,
in incorporating the findings of others,
in disseminating our observations,
in extending (generalizing) our experimental observations
- through exploratory discovery and modeling -
in directions completely unanticipated

REFERENCES

IGrC papers at: <https://dblp.uni-trier.de/pers/hd/s/Skowron:Andrzej>

J. Bocheński, *A history of formal logic*. University of Notre Dame Press 1961

G. Bel-Enguix, M.D. Jiménez-López (eds.): *Language as a Complex System: Interdisciplinary Approaches*. Wiley-Blackwell (2009)

Bower, J.M., Bolouri, H. (Eds.): *Computational Modeling of Genetic and Biochemical Networks*. MIT Press, Cambridge, MA (2001)

F. P. Brooks, Jr.: *The Mythical Man-Month: Essays on Software Engineering, Anniversary Edition*. Addison-Wesley (1995)

E. Y. Chang: *Multi-LLM Agent Collaborative Intelligence: The Path to Artificial General Intelligence*. Association for Computing Machinery, New York, NY 2025. doi.org/10.1145/3749421

J. E. Delanois, S. Joshi et al.: *Context is All You Need* (2026). doi.org/10.48550/arXiv.2604.04364

Dourish, P.: *Where the Action Is. The Foundations of Embodied Interaction*. The MIT Press (2004)

D. Deutsch, A. Ekert, R. Lupacchini, *Machines, logic and quantum physics*. Bull. Symbolic Logic 6, 265–283 (2000) doi.org/ 10.2307/421056

S. Dutta, A. Skowron: *Interactive granular computing model for intelligent systems*. ICIS 2020 doi.org/10.1007/978-3-030-74826-5_4

S. Dutta, A. Skowron: *Toward a computing model dealing with complex phenomena: Interactive granular computing*. ICCCI 2021. doi.org/10.1007/978-3-030-88081-1_1

A. Ehrenfeucht, G. Rozenberg: *Reaction Systems: A Model of Computation Inspired by Biochemistry*, LNCS 6224, 1–3 (2010). doi.org/10.1007/978-3-642-14455-4_1

Angela D. Friederici: *Language in Our Brain: The Origins of a Uniquely Human Capacity*. MIT Press 2017

N. C. Ellis, D. Larsen-Freeman: *Title: Language as a Complex Adaptive System*. Wiley-Blackwell (2009)

J. Fang et al.: *A Comprehensive Survey of Self-Evolving AI Agents: A New Paradigm Bridging Foundation Models and Lifelong Agentic Systems* (2025). doi: 10.48550/arXiv.2508.07407

C. Gershenson, F. Heylighen.: *How can we think the complex?* In: Richardson, K. (Ed.): *Managing Organizational Complexity: Philosophy, Theory and Application*, pp. 47–61. Information Age Publishing (2005)

Y. Gurevich: *What's an algorithm?* (2011). <https://www.youtube.com/watch?v=FX2J24u92GI>

R. Harper: *The Shape of Thought. Reasoning in the Age of AI*. McGill-Queen's University Press (2025)

R. Hightower: *Chapter 9: Leveraging Advanced Reasoning Models*. In: *Programming the OpenAI APIs with Python: A Comprehensive Guide to Building AI Applications*. <https://rick-hightower.notion.site/Chapter-9-Leveraging-Advanced-Reasoning-Models-1e0d6bbdbbea808190d0e3fa0f26192c?pvs=21>

J. R. Hobbs: *Granularity*. In: *Proceedings of the Ninth International Joint Conference on Artificial Intelligence (IJCAI 1985)*, pp. 432–435 (1985). URL <https://www.ijcai.org/proceedings/1985-1>

J. H. Holland: *Signals and Boundaries. Building Blocks for Complex Adaptive Systems*. MIT Press (2014). doi.org/10.7551/mitpress/9412.001.0001

E. Ippoliti, F. Sterpetti (eds.): *The Heuristic View. Logic, Mathematics, and Science*. Springer (2025). doi.org/10.1007/978-3-031-94709-4

A. Jankowski: *Interactive Granular Computations in Networks and Systems Engineering: A Practical Perspective*. Springer (2017). doi.org/10.1007/978-3-319-57627-5

Keefe, R. (2000) *Theories of Vagueness*. Cambridge Studies in Philosophy, Cambridge, UK)

Kuai, H., Huang, J.X., Tao, X. et al. *Web Intelligence (WI) 3.0: in search of a better-connected world to create a future intelligent society*. *Artif Intell Rev* 58, 265 (2025). doi: [10.1007/s10462-025-11203-z](https://doi.org/10.1007/s10462-025-11203-z)

Long et al. : *A Survey: Learning Embodied Intelligence from Physical Simulators and World Models*. doi.org/10.48550/arXiv.2507.00917

L. Ma, M. Li: Covering rough set models, fuzzy rough set models and soft rough set models induced by covering similarity. *Information Sciences* 689 (2025) 121520
doi.org/10.1016/j.ins.2024.121520

A. H. Maslow: A theory of human motivation. *Psychological Review* 50: 370-396 (1943)

M. Mitchell: Abstraction and Analogy-Making in Artificial Intelligence, *Annals Reports of the New York Academy of Sciences* 1505(1), 79-101 (2021)

R. M. Monarch, *Human-in-the-Loop Machine Learning. Active Learning and Annotation for Human-Centered AI*, MANNING, Shelter Island, NY (2021)

Kuai, H., Huang, J.X., Tao, X. et al. Web Intelligence (WI) 3.0: in search of a better-connected world to create a future intelligent society. *Artif Intell Rev* 58, 265 (2025). doi: [10.1007/s10462-025-11203-z](https://doi.org/10.1007/s10462-025-11203-z)

Long et al. : A Survey: Learning Embodied Intelligence from Physical Simulators and World Models. doi.org/10.48550/arXiv.2507.00917

L. Ma, M. Li: Covering rough set models, fuzzy rough set models and soft rough set models induced by covering similarity. *Information Sciences* 689 (2025) 121520
doi.org/10.1016/j.ins.2024.121520

R. Mariani et al.: Trustworthy AI. Part I. *Computer* 56(2) 14-18 (February 2023).
<https://doi.org/10.1109/MC.2022.3227683>

A. H. Maslow: A theory of human motivation. *Psychological Review* 50: 370-396 (1943)

M. Mitchell: Abstraction and Analogy-Making in Artificial Intelligence, *Annals Reports of the New York Academy of Sciences* 1505(1), 79-101 (2021)

R. M. Monarch, *Human-in-the-Loop Machine Learning. Active Learning and Annotation for Human-Centered AI*, MANNING, Shelter Island, NY (2021)

U. Nisa et al.: *Agentic AI: The age of reasoning—A review*. *Journal of Automation and Intelligence* (2025). doi: [10.1016/j.jai.2025.08.003](https://doi.org/10.1016/j.jai.2025.08.003)

A. Noë: *Action in Perception*, MIT Press (2004)

E.R. Norwitz, G. Rubin, F. Reed-Tsochas: *Healthcare as a Complex Adaptive System: Understanding Health Disparity as an Emergent Phenomenon*. *Curr Obstet Gynecol Rep* 14, 20 (2025). doi: [10.1007/s13669-025-00425-1](https://doi.org/10.1007/s13669-025-00425-1)

C. L. Ortiz Jr.: *Why we need a physically embodied Turing test and what it might look like*, *AI Magazine* 37, 55–62 (2016). doi.org/[10.1609/aimag.v37i1.2645](https://doi.org/10.1609/aimag.v37i1.2645)

Z. Pawlak: *Rough sets. Theoretical Aspects of Reasoning About Data*. Kluwer (1991). doi.org/[10.1007/978-94-011-3534-4](https://doi.org/10.1007/978-94-011-3534-4)

Z. Pawlak, A. Skowron: *Rough Sets and Boolean Reasoning*. *Inf. Sci.* 177(1), 41-73 (2007). doi.org/[10.1016/j.ins.2006.06.007](https://doi.org/10.1016/j.ins.2006.06.007)

Paavola, S.: *Abduction as a Logic of Discovery*. In: Magnani, L. (eds) *Handbook of Abductive Cognition* (pp. 43-60). Springer, Cham (2023). doi: [10.1007/978-3-030-68436-5_9-1](https://doi.org/10.1007/978-3-030-68436-5_9-1)

W. Pedrycz: *Granular Computing. Analysis and Design of Intelligent Systems*. CRC Press, Taylor & Francis (2013). doi.org/[10.1201/9781315216737](https://doi.org/10.1201/9781315216737)

M. Remondino: *Reactive and deliberative agents applied to simulation of socio-economical and biological systems*. *International Journal of Simulation* 6(12-13) 11-25 (2005)

B. Shneiderman, *Human Centered AI*, Oxford University Press, Oxford, UK (2022).doi.org/[10.1093/oso/9780192845290.001.0001](https://doi.org/10.1093/oso/9780192845290.001.0001)

A.Skowron. A. Jankowski: *Rough Sets and Interactive Granular Computing*. *Fundamenta Informaticae* 147 (2-3), 371–385 (2016). doi.org/[10.3233/FI-2016-1413](https://doi.org/10.3233/FI-2016-1413)

A. Skowron, A. Jankowski, and S. Dutta: *Interactive Granular Computing*. *Granular Computing* 1, 95–113 (2016). doi.org/[10.1007/s41066-015-0002-1](https://doi.org/10.1007/s41066-015-0002-1)

A. Skowron, A. Jankowski, S. Dutta: *Interactive Granular Computing: Toward Computing Model for Complex Intelligent Systems* (invited contribution). In: M. Bolanowski, M. Ganzha, L. Maciaszek, M. Paprzycki, D. Ślęzak (eds.), *Proceedings of the 20th Conference on Computer Science and Intelligence Systems (FedCSIS), Annals of Computer Science and Information Systems (ACSIS)*, Vol. 43, pp. 59–72 (2025) doi: [10.15439/2025F6355](https://doi.org/10.15439/2025F6355)

A. Skowron, D. Ślęzak: *Rough Sets in Interactive Granular Computing: Toward Foundations for Intelligent Systems Interacting with Human Experts and Complex Phenomena*. IJCRS 2023. doi.org/10.1007/978-3-031-50959-9

A. Skowron, Z. Suraj (1995). *Discovery of concurrent data models from experimental data tables: A rough set approach*, *Proceedings of the First International Conference on Knowledge Discovery and Data Mining*, Montreal, August, 1995, AAAI Press, Menlo Park CA 1995, 288-293

L. P. Thiele: *The Heart of Judgment Practical Wisdom, Neuroscience, and Narrative*. Cambridge University Press 2006

L. Valiant: *Probably Approximately Correct. Nature's Algorithms for Learning and Prospering in a Complex World*. Basic Books, A Member of the Perseus Books Group, New York (2013)

L. Valiant: <http://people.seas.harvard.edu/~valiant/researchinterests.htm>

V. Vapnik, *Statistical Learning Theory*, Wiley (1998)

Wang, P. et al.: *AI-driven complex systems redefine cognitive science*. *The Innovation* 7(5): 101314, May 4, 2026

Xu, J.: *Biological Computing*. Springer, Singapore (2025). doi: [10.1007/978-981-96-3870-3](https://doi.org/10.1007/978-981-96-3870-3)

Xu, J. (2025). *Protein Computing*. In: *Biological Computing*. Springer, Singapore (2025). doi: [10.1007/978-981-96-3870-3_12](https://doi.org/10.1007/978-981-96-3870-3_12)

L. A. Zadeh: *From computing with numbers to computing with words – From manipulation of measurements to manipulation of perceptions*. *IEEE Transactions on Circuits and Systems* 45(1), 105–119 (1999). doi: [10.1109/81.739259](https://doi.org/10.1109/81.739259)

L. A. Zadeh: Foreword. In: S.K. Pal, L. Polkowski, A. Skowron (eds.) *Rough-Neural Computing. Techniques for Computing with Words*. Springer 2004.

W. Zaremba, OpenAI (5 steps of development),
<https://www.youtube.com/watch?v=pX0BZwFEPiE>

B. Zeng, M. Liu, Z. Zhen: GraphMASAL: A Graph-based Multi-Agent System for Adaptive Learning. *Proc. of the 25th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2026)*, C. Amato, L. Dennis, V. Mascardi, J. Thangarajah (eds.), May 25 – 29, 2026, Paphos, Cyprus. doi: 10.48550/ARXIV.2511.11035

H. Zhang et al.: Human-machine collaboration in reshaping visual perception and cognition. *The Innovation* 7(3): 101255, March 2, 2026

Q. Zhang et al.: Agentic Context Engineering: Evolving Contexts for Self-Improving Language Models. (2026). doi: 10.48550/arXiv.2510.04618

Q. Zheng et al: Machine Memory Intelligence: Inspired by Human Memory Mechanisms. *Engineering* 55: 24-35 (2025). doi: 10.1016/j.eng.2025.01.012

THANK YOU!